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A PROJECTION SPACE MAP METHOD FOR LIMITED ANGLE RECONSTRUCTION

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Abstract

We present a method to reconstruct images from finite sets of noisy projections which are available only over limited or sparse angles. The method solves a constrained optimization problem to find a maximum *a posteriori* (MAP) estimate of the *full* 2-D Radon transform of the object, using prior knowledge of object mass, center of mass, and convex support, and information about fundamental constraints and smoothness of the Radon transform. This efficient primal-dual algorithm consists of an iterative local relaxation stage which solves a partial differential equation in Radon-space, followed by a simple Lagrange multiplier update stage. The object is reconstructed using convolution backprojection applied to the Radon transform estimate.

I. Introduction

Although limited angle tomography has been widely discussed in the literature, adequate imagery is still not obtainable in discplines in which there are both restricted viewing angles and low signal to noise ratios (cf. [1] and references). The problem is fundamentally one of inverting the 2-D Radon transform given by

$$g(t,\theta) = \mathcal{R}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\delta(t-\omega^{\mathrm{T}}x)\,dx\,, \qquad (1)$$

where $f(x)$ is a real function defined on the plane (which we will assume to be zero outside the disk of radius T centered at the origin) and $\omega = [\cos\theta \;\; \sin\theta]^{\mathrm{T}}$. Thus, the 2-D Radon transform $g(t,\theta)$, for fixed t and θ, is a *line integral* of the function $f(x)$ along the line with lateral displacement t and unit normal ω.

When one obtains a large number of accurate measurements of $g(t,\theta)$ for $t \in [-T,T]$ and $\theta \in [0,\pi)$, then a high-quality reconconstruction of $f(x)$ may be made using conventional techniques, e.g., convolution backprojection [2]. However, when the line integrals are observed in noise, and when the angular range is restricted to a subset of $[0,\pi)$ — i.e. either the limited- or sparse-angle situation — then these conventional techniques are not adequate. Some of the methods in the literature designed to account for the limited- and sparse-angle cases, and in some cases the noise, include modified transform methods, iteration between spaces, and finite series expansion methods (see [1] and references). The methods most closely related to our methods are those which seek to *directly* estimate the *full* Radon transform such as in [3] and [4].

II. Consistency and Support

Certain mathematical properties of the 2-D Radon tranform are used to advantage in our reconstruction method. The first property is one of consistency: *not all functions $g(t,\theta)$ are Radon transforms of some function $f(x)$.* A full discussion of the consistency conditions required of a 2-D Radon transform may be found in [5]. What we require in this paper is the periodicity condition given by $g(t,\theta) = g(-t,\theta+\pi)$, and the two moment constraints given by

$$\int_{-T}^{T} g(t,\theta)\,dt = m\,, \qquad (2)$$

and

$$\frac{1}{m}\int_{-T}^{T} t g(t,\theta)\,dt = c(\theta)\,, \qquad (3)$$

where $c(\theta)$ is a cosinusoidal function in θ. Both m and $c(\theta)$ may often be estimated quite accurately [6],[7], so that we may use these two equations as *constraints* on the full Radon transform to be estimated. We assume in what follows that a pre-processing stage scales and shifts the measurements so that $m=1$ and $c(\theta)=0$.

The second mathematical property of the 2-D Radon transform is one of support: *the convex hull of the support $\mathcal{F}$ of the function $f(x)$ has a one-to-one correspondence to the support $\mathcal{G}$ of $\mathcal{R}\{f(x)\}$*, where by support we mean the set of points where the function is non-zero. Therefore, if we knew hul($\mathcal{F}$) *a priori*, we would insist that any estimate of $g(t,\theta)$ be zero for $(t,\theta) \notin \mathcal{G}$. Our approach, instead, assumes that we have only an *estimate* of hul($\mathcal{F}$) (produced

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perhaps by the methods in [7]), and therefore that $g(t,\theta)$ should be *small* where $(t,\theta) \notin \mathcal{G}$.

III. Variational Formulation

Consider the problem, which we refer to as (V), to minimize

$$\begin{aligned} I &= \iint_{\mathcal{Y}_O} \frac{1}{2\sigma^2}(y-g)^2\,dt\,d\theta + \iint_{\bar{\mathcal{G}}} \kappa g^2\,dt\,d\theta \\ &+ \iint_{\mathcal{Y}_T} \left[\beta\left(\frac{\partial g}{\partial t}\right)^2 + \gamma\left(\frac{\partial g}{\partial \theta}\right)^2\right] dt\,d\theta \qquad (4) \end{aligned}$$

subject to the equality constraints given by (2) and (3) and boundary conditions $g(T,\theta) = g(-T,\theta) = 0$ and $g(t,0) = g(-t,\pi)$ where κ, β, and γ are positive constants. Here, $\mathcal{Y}_T = \{(t,\theta) \mid -T \le t \le T, 0 \le \theta \le \pi\}$ and $\mathcal{Y}_O$ is a subset of $\mathcal{Y}_T$ over which (noisy) measurements y are available, and $\bar{\mathcal{G}} = \mathcal{Y}_T - \mathcal{G}$.

The first term in I represents a penalty which seeks to keep the estimate close to the observations. The second term is a penalty for non-zero values outside the support of the Radon transform, and finally, the third term penalizes large derivatives in both the vertical and horizontal direction, and is therefore a smoothing term.

A necessary and sufficient condition for $g(t,\theta)$ to be a solution to (V) is that is satisfy the following second order partial differential equation (PDE) [7]

$$\begin{aligned} \left(2\kappa\bar{\chi}_G + \frac{1}{\sigma^2}\chi_Y\right) g &- 2\beta\frac{\partial^2 g}{\partial t^2} - 2\gamma\frac{\partial^2 g}{\partial \theta^2} \\ &= \frac{1}{\sigma^2}\chi_Y y - \lambda_1(\theta) - \lambda_2(\theta) t \qquad (5) \end{aligned}$$

and the additional boundary condition $\partial g(t,0)/\partial t = \partial g(-t,\pi)/\partial t$, where $\bar{\chi}_G$ and χ_Y are the indicator functions for $\bar{\mathcal{G}}$ and $\mathcal{Y}_O$, respectively. In addition, $g(t,\theta)$ must satisfy the original constraints and boundary conditions. It is important to note that (5) contains three unknown functions: $g(t,\theta)$, and two Lagrange multiplier functions $\lambda_1(\theta)$ and $\lambda_2(\theta)$ (one for each constraint).

The numerical solution to (5), which we describe below, is found on a discrete lattice system in $\mathcal{Y}_T$. It turns out that this solution, which seeks of a finite number of variables denoted by the vector g, is *exactly* the maximum *a posteriori* (MAP) estimate of g, when g is described by a certain Markov random field prior probability, and when the noise is given by additive independent, zero-mean Gaussian random variables with variance σ^2 [7].

IV. Local Relaxation Algorithm

To solve (5) we must find both $g(t,\theta)$ and the two Lagrange multiplier functions, $\lambda_1(\theta)$ and $\lambda_2(\theta)$, so that the PDE itself is satisfied and the mass and center of mass constraints are satisfied as well. Since for *fixed* $\lambda_1(\theta)$ and $\lambda_2(\theta)$, the PDE is elliptic in $g(t,\theta)$, we may solve it numerically on a discrete lattice system. This suggests a primal-dual approach where we solve the PDE in the primal stage for fixed λ_1 and λ_2, followed by a dual stage which updates λ_1 and λ_2. We use a very efficient local relaxation algorithm (which may be implemented in parallel) due to Kuo et. al. [8] to solve the PDE in the primal phase, and a simple Lagrange multiplier update stage (see [9]). Fortunately, the value of the *final* Lagrange multipliers may often be estimated to high accuracy before beginning the iteration, which speeds up converegence dramatically [7]. We summarize the algorithm below.

Local Relaxation Algorithm:

1. Estimate final Lagrange multipliers $\lambda_1^*(\theta)$ and $\lambda_2^*(\theta)$.
2. Set $\lambda_1^0(\theta) = \hat{\lambda}_1^*(\theta)$ and $\lambda_2^0(\theta) = \hat{\lambda}_2^*(\theta)$.
3. Set $k = 1$ and $g^0 = y$.
4. Solve PDE numerically to yield g^k.
5. Does g^k satisfy the constraints?
6. If not, update Lagrange multipliers according to

$$\lambda_1^{k+1}(\theta) = \lambda_1^k + \alpha\left(m - \int_{-T}^{T} g^k(t,\theta)\,dt\right)$$

$$\lambda_2^{k+1}(\theta) = \lambda_2^k + \alpha\left(0 - \int_{-T}^{T} t g^k(t,\theta)\,dt\right)$$

Set $k \leftarrow k+1$ and goto 4.

7. Otherwise, we are done and $\hat{g} = g^k$.

This algorithm converges to the globally optimum solution provided that α is chosen small enough [9].

V. Experimental Results

In this section, we present the results of two experiments, designed to show the overall performance of the algorithm on a limited-angle case and on a sparse-angle case. The object that is used in these simulations is an ellipse with the letters M I T in its interior, shown in Fig. 1 using an 81 by 81 discretization. Fig. 2 shows a noisy sinogram (SNR=10.0dB), consisting of 81 rows (sampling t) and 60 columns (sampling θ), created by adding independent samples of zero-mean Gaussian noise with variance σ^2 to each element of the true sinogram (not shown).

Fig. 3 shows an object reconstruction using convolution backprojection (CBP) in which only the first 40 of 60 (leftmost) projections of the sinogram in Fig. 2 were used. A reconstruction obtained after processing using the local relaxation MAP algorithm described in Section IV is shown in Fig. 4. In this case, the support $\mathcal{G}$ and the mass m of the Radon tranform were *estimated* using methods described in [7] and [10], while the center of mass was

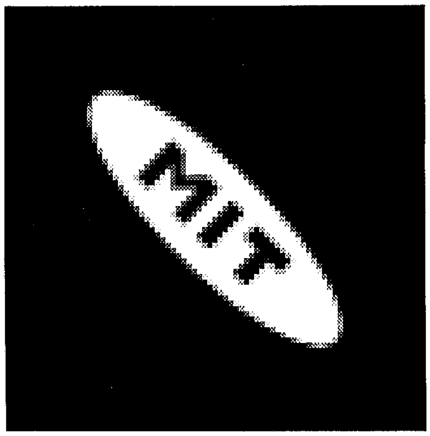

Fig. 1. Original MIT ellipse.

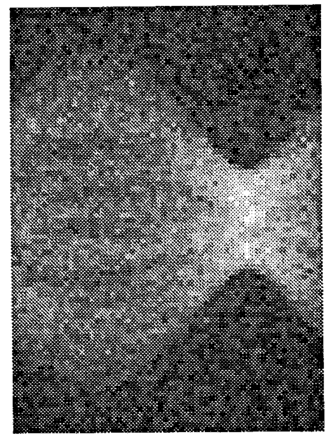

Fig. 2. 10dB sinogram of MIT ellipse.

(correctly) assumed to be zero. The coefficients κ, γ, and β were set to 5.0, 0.05, and 0.01, respectively.

Fig. 5 shows an object reconstruction using convolution backprojection (CBP) in which only 10 evenly spaced projections of the sinogram in Fig. 2 were used. A reconstruction obtained after processing using the same coefficients as above is shown in Fig. 6.

One can see from these two experiments a dramatic improvement in the reconstructions. The limited-angle case shown in Figs. 3 and 4 shows most clearly how support information — which was *estimated* from measurements in this case — can improve the definition of the object boundaries. The sparse-angle case shows considerable improvement resulting primarily from the horizontal smoothing effects and constraints. The intermediate result (not shown) in each case is a smoothed, interpolated, and feasible (with respect to the mass and center of mass constraints) sinogram.

VI. Discussion

We have demonstrated in this paper a method based on estimation principles for reconstructing images from their noisy and limited-angle or sparse-angle Radon transforms. We have shown that including certain types of prior knowledge can lead to improved reconstruction over convolution back-projection applied directly to the measurements. A hierarchical algorithm described in [7], however, allows much of this information to be estimated in previous stages; therefore, the method is largely self-contained. Many extensions to this work are possible. One extension which we have explored in [7] is to incorporate more than just *two* of the constraints inherent to the Radon transform.

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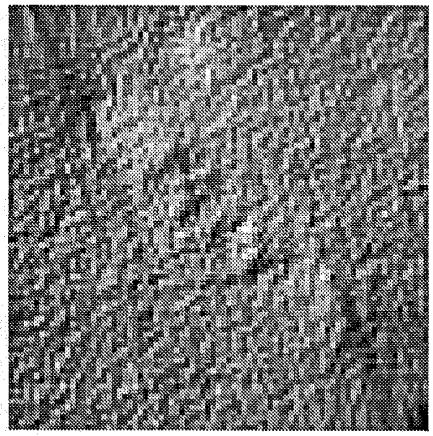

Fig. 3. Limited-angle reconstruction using CBP.

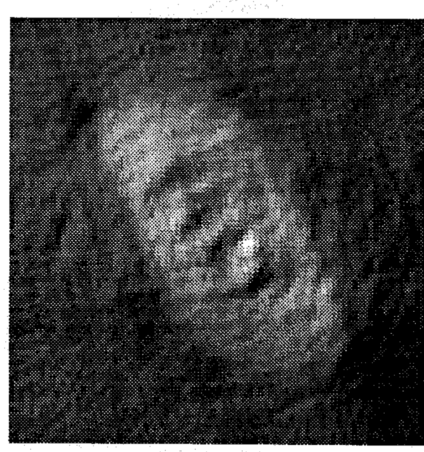

Fig. 4. Limited-angle reconstruction after processing.

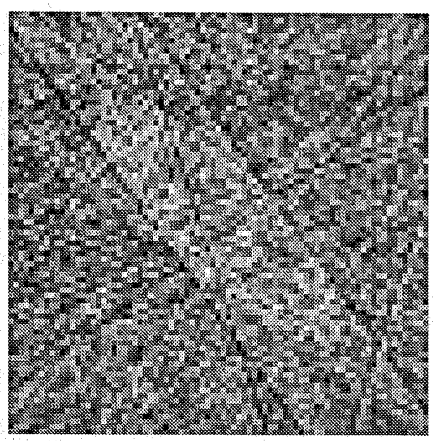

Fig. 5. Sparse-angle reconstruction using CBP.

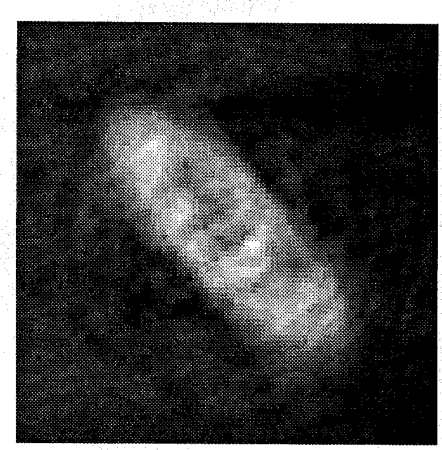

Fig. 6. Sparse-angle reconstruction after processing.

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