CONSISTENCY AND CONVEXITY IN OBJECT RECONSTRUCTION FROM PROJECTIONS

Jerry L. Prince

Image Analysis and Communications Laboratory
Department of Electrical and Computer Engineering,
The Johns Hopkins University, Baltimore, MD 21218

Abstract

We present a projection-space approach for reconstruction from projections. This approach uses the known convex support of the object both as a penalty term in a variational problem defined in projection-space and as a guide to the specification of optimal smoothing coefficients that may vary spatially. Partial consistency of the sinogram is provided by including mass and center-of mass constraints in the variational formulation. We provide an outline of the general approach and calculations for a specific example. Computer simulations are provided for evaluation of the performance in this special case.

I. Introduction

In many tomography applications, measured projections may be noisy or incomplete or both [1]. In these cases, the already ill-posed reconstruction problem becomes even worse, and regularization must be used [2]. Regularization, or imposition of prior knowledge, is used to provide either a unique solution, or at least one that meets all the constraints implied by the available prior knowledge. In this paper, we examine two commonly used constraints: 1) that the projections must be consistent with some object and 2) that the object lies within a known convex region of support. We also assume the object to be reasonably constant over its convex support, although other assumptions about the object profile may be made also.

The requirement of consistent projections has been dealt with in two ways in the past. In the first approach, one has in hand a method to perform the forward problem on test objects — i.e., reprojection — and seeks objects whose reprojections match the data in some defined sense. In the second, one uses a mathematical statement of consistent projections and seeks projections that are mathematically consistent. The object is then reconstructed using ordinary methods such as convolution backprojection, the Fourier method, or algebraic reconstruction techniques [3,4]. The former approach is an *iteration-between-spaces* approach while the latter is a *projection-space* approach.

Iteration-between-spaces approaches have the advantage that constraints and other information may be applied in either domain (object or projection) with ease, with the disadvantage that optimality in the object domain is difficult to maintain since the noise structure is quite complicated. Furthermore, as the iteration proceeds numerical error (in reprojection) often increases. Projection-space approaches have the disadvantage that it is difficult to apply information about the object in the projection space, but the advantage that the noise structure is easily defined and thus optimality is guaranteed.

In previous work [5,6] we have reported a projection-space approach which applies information about the convex support of the object, consistency of projections, and smoothness of sinograms[1] as regularization. The approach solves a variational problem defined on sinograms which results in a partial differential equation (PDE) that is solved numerically. It was shown that the numerical solution of this problem is equivalent to that which would solve a maximum a posteriori (MAP) estimation problem for sinograms that are characterized *a priori* by a certain Markov random field (MRF) [6].

In this paper, we show that by slightly increasing the generality of the variational objective function, we can specify a variational formulation (and hence an implied Markov random field model) which has a specific object as its most likely prior object. We solve this problem for the special case of the disk of radius R with constant value in the interior. Computer simulations are shown for this special case and we provide a discussion of the more general approach to arbitrary convex objects.

II. Projection-Space Approach

The 2-D Radon transform of an object $f(x)$ is given by

$$g(t,\theta) = \int_{x\in\mathbb{R}^2} f(x)\delta(t-\omega\cdot x)\,dx\,, \tag{1}$$

where $\omega = (\cos\theta, \sin\theta)$ is the unit vector orthogonal to the lines of integration and $\delta(\cdot)$ is the Dirac delta function. Viewed as a function of t for fixed θ, $g(t,\theta)$ is called a *projection* of $f(x)$.

The 2-D Radon transform obeys the periodicity condition $g(-t,\theta) = g(t,\theta+\pi)$ and the integral constraints

$$\int_{-\infty}^{\infty} g(t,\theta)\,dt = \mu\,, \tag{2}$$

$$\frac{1}{\mu}\int_{-\infty}^{\infty} g(t,\theta)t\,dt = c(\theta)\,, \tag{3}$$

where $c(\theta) = a\cos\theta + b\sin\theta$ for some real numbers a and b. These two equations are known as the *mass* and *center-of-mass*

[1] A sinogram is an image of the projections.

CH2872-0/90/0000-0543 © $1.00 1990 IEEE

constraints, respectively. They describe the two lowest order consistency conditions prescribed by the Ludwig-Helgason conditions [7,8]. In this paper we consider a sinogram to be consistent if it satisfies these conditions; we also insist on knowing μ and that $a = b = 0$. In another paper [5], we describe an approach that considers higher moments without requiring this specific prior knowledge.

Restricting our attention to objects $f(x)$ that are zero outside a disk of radius T centered at the origin, we see that $g(t,\theta) = 0$ for $|t| > T$. Hence, a 2-D Radon transform is completely characterized by the values of $g(t,\theta)$ over the domain $\mathcal{Y} = \{(t,\theta) \mid -T \le t \le T, 0 \le \theta \le \pi\}$. We call $\mathcal{Y}$ the *sinogram domain*, and an image of $g(t,\theta)$ over this domain is called a *sinogram*.

Measurements are given by $y(t,\theta) = g(t,\theta) + w(t,\theta)$ where $w(t,\theta)$ is a zero-mean white Gaussian random process with noise intensity σ^2. The measurements are presumed to be available only over a subset of projections $\mathcal{Y}_o \subset \mathcal{Y}$. We then define the sinogram estimate to be the sinogram that minimizes

$$\begin{aligned} I &= \iint_{\mathcal{Y}_o} \frac{1}{2\sigma^2}(y-g)^2\, dt\, d\theta + \iint_{\bar{\mathcal{G}}} \kappa g^2\, dt\, d\theta \\ &+ \iint_{\mathcal{Y}} \left[\beta(t,\theta)\left(\frac{\partial g}{\partial t}\right)^2 + \gamma(t,\theta)\left(\frac{\partial g}{\partial \theta}\right)^2\right] dt\, d\theta\,, \end{aligned} \tag{4}$$

subject to the equality constraints

$$\begin{aligned} J_1 &= \mu = \int_{-T}^{T} g(t,\theta)\, dt\,, \quad \text{and} \\ J_2 &= 0 = \frac{1}{\mu}\int_{-T}^{T} g(t,\theta) t\, dt\,. \end{aligned} \tag{5}$$

In addition, the boundary conditions

$$\begin{aligned} g(T,\theta) &= g(-T,\theta) = 0 \quad \text{and} \\ g(t,0) &= g(-t,\pi) \end{aligned} \tag{6}$$

must also be satisfied. In (4) the region $\bar{\mathcal{G}}$ over which the second integral is taken is the complement of the region $\mathcal{G}$ in the sinogram, which corresponds to the convex support of the object. This term allows one to include information pertaining to the convex support of the object directly in projection-space.

The difference between this formulation and that given in [5] and [6] is that here, the positive coefficients multiplying the squares of the two partial derivatives of $g(t,\theta)$ are allowed to vary with t and θ. In the previous work, these coefficients were constant and this was justified on the heuristic grounds that objects tend to have line integrals that are nearly the same when displaced either in lateral position or rotational angle. In the following section we present an approach to the selection of $\beta(t,\theta)$ and $\gamma(t,\theta)$ that has a precise definition, although it may be difficult to solve in the general case.

III. Inverse Variational Problem

If one knows the convex support of the object, this information can be directly included in the variational formulation described above. However, it is reasonable that this information should also affect how one selects the smoothing coefficients $\gamma(t,\theta)$ and $\beta(t,\theta)$. In this section we explore the criterion that these coefficients should be chosen so that a *particular specified object* has a sinogram that minimizes I with $\mathcal{Y}_o = \emptyset$, subject to the given equality constraints and a new boundary condition provided by the convex support. We will develop the general approach, calculate a specific example, and present results to demonstrate the performance on a simulated object for this special case.

As a specific example of our approach, suppose that within the convex support of the object we expect (*a priori*) that the object will be reasonably constant. We now seek $\gamma(t,\theta)$ and $\beta(t,\theta)$ so that the sinogram $g^\star(t,\theta)$ of this constant object minimizes I in the absence of observations, and with the boundary conditions such that $g(t,\theta) = 0$ when $(t,\theta) \in \bar{\mathcal{G}}$. The second integral of I is zero since $g^\star(t,\theta) = 0$ when $(t,\theta) \in \bar{\mathcal{G}}$. Therefore, the goal is to find $\gamma(t,\theta) > 0$ and $\beta(t,\theta) > 0$ such that $g^\star(t,\theta)$ minimizes

$$I_2 = \iint_{\mathcal{Y}} \left[\beta(t,\theta)\left(\frac{\partial g}{\partial t}\right)^2 + \gamma(t,\theta)\left(\frac{\partial g}{\partial \theta}\right)^2\right] dt\, d\theta \tag{7}$$

subject also to the constraints in (5) and the new boundary condition provided by $\bar{\mathcal{G}}$. We refer to this variational problem as (V2).

It is tempting at this point to set $g(t,\theta) = g^\star(t,\theta)$ and find $\gamma(t,\theta)$ and $\beta(t,\theta)$ that minimize I_2, but this is wrong. We really face an inverse problem here, that of deducing what coefficients in the variational problem would give rise to the stationary function $g^\star(t,\theta)$. We proceed by finding the necessary conditions for $g(t,\theta)$ to minimize (or maximize) I_2 and then substitute in $g^\star(t,\theta)$ to see what equation $\beta(t,\theta)$ and $\gamma(t,\theta)$ must satisfy. There is no reason to expect a unique answer, although the form of the variational problem does restrict the class of solutions dramatically.

The General Case

In the general case, the stationary function $g(t,\theta)$ must satisfy the Euler-Lagrange equation (cf. [9])

$$\frac{\partial}{\partial t}\left[2\beta(t,\theta)g_t(t,\theta)\right] + \frac{\partial}{\partial \theta}\left[2\gamma(t,\theta)g_\theta(t,\theta)\right] - \lambda_1(\theta) - \lambda_2(\theta)t = 0 \tag{8}$$

where $g_t(t,\theta)$ and $g_\theta(t,\theta)$ denote the partial derivatives of $g(t,\theta)$ with respect to t and θ, respectively. The two functions $\lambda_1(\theta)$ and $\lambda_2(\theta)$ are unknown Lagrange multipliers, corresponding to the two constraints, which may vary with θ, in general. After substituting $g^\star(t,\theta)$ for $g(t,\theta)$ in (8), we find that the resulting equation is a first order PDE in both $\beta(t,\theta)$ and $\gamma(t,\theta)$ and can be solved by any number of techniques, analytically or numerically. We do not explore the general case any further in this paper, but instead focus on a specific case which we can easily solve and contrast to the case of constant $\beta(t,\theta)$ and $\gamma(t,\theta)$.

A Specific Case

Suppose the convex support is a disk of radius R centered at the origin, and suppose that the true object is expected to be reasonably constant in the interior and has mass μ. We now find $\beta(t,\theta)$ and $\gamma(t,\theta)$ so that the object that is *exactly* constant on this disk, has mass μ, and is zero outside the disk has a sinogram which is a stationary function of $(V2)$.

Because of the rotational symmetry $g^\star(t,\theta)$ does not depend on θ, and since the boundary conditions are periodic in θ, we may select $\gamma(t,\theta) = \gamma_0$, a constant. Also, since $g^\star_\theta(t,\theta) = 0$, and $\lambda_1(\theta)$

and $\lambda_2(\theta)$ do not depend on θ in this case, the Euler-Lagrange equation now depends only on the variable t. To simplify the following we rewrite the Euler-Lagrange equation as

$$\frac{\partial}{\partial t}\left(2\beta(t)g_t(t)\right) - \lambda_1 - \lambda_2 t = 0\,. \tag{9}$$

Then, after integrating and rearranging, we find that

$$\beta(t) = \frac{2\lambda_1 + \lambda_2 t^2 + 2C}{4g_t(t)}\,, \tag{10}$$

where C is a constant of integration.

The disk of radius R centered at the origin with mass μ has the projection (at any angle)

$$g^*(t) = \begin{cases} \frac{2\mu}{\pi R^2}\sqrt{R^2 - t^2} & -R \le t \le R \\ 0 & \text{otherwise} \end{cases}\,. \tag{11}$$

Considering on the region of convex support $|t| \le R$, taking the derivative, and substituting the result into (10), yields after simplification

$$\beta(t) = -\frac{\pi R^2}{8\mu}\sqrt{R^2 - t^2}\left(2\lambda_1 + \lambda_2 t + \frac{2C}{t}\right)\,. \tag{12}$$

Since $\beta(t)$ must be nonnegative for $t \in [-R, R]$ it must be true that $\lambda_2 = C = 0$. Setting $\lambda_1 = -4\beta_0\mu/\pi R^3$ (which is arbitrary) yields

$$\beta(t) = \frac{\beta_0}{R}\sqrt{R^2 - t^2}\,, \tag{13}$$

which turns out to be proportional to the projection $g^*(t)$. However, this is a coincidence and not the general rule as we see in the following development.

The Constant Coefficient Case

We now have that the constant disk of radius R, centered at the origin and with mass μ, has a sinogram which is a stationary function of the variational problem (V2) for $\gamma(t,\theta)$ constant and $\beta(t,\theta)$ as in (13). A natural question to ask is what is the sinogram that solves (V2) when both $\gamma(t,\theta)$ and $\beta(t,\theta)$ are constant, as was the case in previous work we have done [5,6].

Proceeding as above, we see that a constant $\gamma(t,\theta)$ together with periodic boundaries in θ assures a solution that has the same projection at all angles and is an even function of t. However, the boundary condition $g(T,\theta) = g(-T,\theta) = 0$ together with the mass constraint and center-of-mass constraints will not generate the projection of a constant object over the disk of radius T centered at the origin. In fact, solving the forward variational problem produces the projection

$$g(t) = \frac{3\mu}{4T}\left(1 - \left(\frac{t}{T}\right)^2\right)\,. \tag{14}$$

The two projections (11) and (14) are compared in Fig. 1 for $T = R = \mu = 1$. The solid line shows (11) and the dashed curve shows (14). One can see by contrasting these projections that the optimal object for $\beta(t,\theta)$ and $\gamma(t,\theta)$ constant has larger magnitude at the origin than near the boundary of the disk. Intuitively, the reason for this is that the constant coefficient model smooths across the boundary, which is set to zero. On the other hand, the varying coefficient model has a smoothing coefficient which goes to zero as it hits the boundary, and therefore does not smooth across it.

In the general case this property of the derived coefficients should become important. For example, given a known convex support, we can force the estimated sinogram to be zero outside this region by setting κ in (4) to be large. However, if one uses constant coefficients, the boundaries will tend to be blurred and the values near the boundaries will be low. In contrast, the derived coefficients effectively decouple the object values from the boundary values. We demonstrate this effect in simulations in the following section.

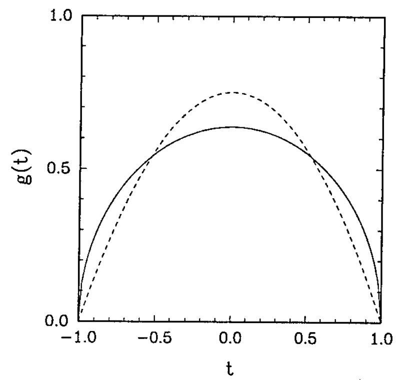
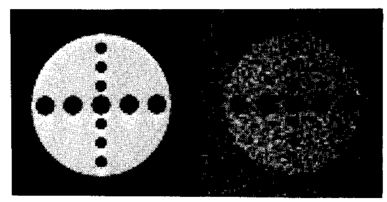


Fig. 1. Optimum prior projections for different variational formulations.



(a) (b)

Fig. 2. (a) Disk object with radius 0.5 and value 1.0. (b) Reconstruction from a 15.0dB noisy sinogram using CBP.

IV. Simulation Results

The disk object shown in Fig. 2a has radius $R = 0.75$, value 1.0, and is centered at the origin. Eleven smaller disks of radii 0.1 and 0.06 are removed from its interior. The object is shown, as are all objects in this paper, using an 81 by 81 pixel image. We obtain 60 true projections evenly spaced in angle with 81 values per projections, also uniformly spaced. Pseudorandom independent zero-mean Gaussian noise is added to each line-integral so the effective signal-to-noise ratio (SNR) is 15.0 dB. A straight reconstruction from this data using convolution backprojection (CBP) is shown in Fig. 2b.

Using the noisy sinogram data, we solve the Euler-Lagrange equation of (8) numerically (see [5]) for the two cases

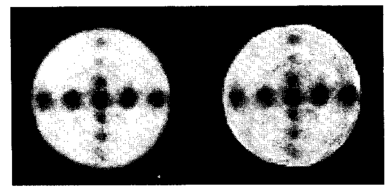
$$\text{Case 1:} \quad \beta(t,\theta) = \beta_0\,, \tag{15}$$

$$\text{Case 2:} \quad \beta(t,\theta) = \frac{\beta_0}{R}\sqrt{R^2 - t^2}\,. \tag{16}$$

Correct support information is used in all simulations with $\kappa = 10,000$, $\beta_0 = 0.1$, and $\gamma_0 = 0.2$. The mass and noise variance were estimated from the data in all cases, however, the center-of-mass was simply presumed to be at the origin (which it is in this case).

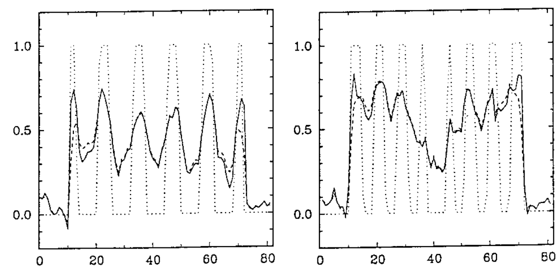
Fig. 3 shows the reconstructed images for the two cases, and Fig. 4 shows horizontal and vertical profiles of each of the resulting images, together with the true profile. One can see in Fig. 3 evidence that the boundary of the object is crisper for Case 2, as we would expect from our development, and we can verify this in the profiles of Fig. 4. The profiles also show the smoothing effect of Case 1 actually propagates well into the object, creating a loss of contrast in the internal features as well.

One disadvantage to the spatially varying coefficients of Case 2 is evident from these experiments. In Fig. 3 one can see that a bright noise spike near the boundary of the object in one of the projections appears as an arc in the image. This is because the spike does not get smoothed much in the t direction since the vertical smoothing coefficient $\beta(t,\theta)$ is approaching zero; the horizontal coefficient is still the same, however, giving rise to this arc. The very fact that we allow the projections to change more rapidly near the disk boundaries to accommodate the more rapidly changing disk shape allows them to also change more rapidly in the event of a noise spike.



(a) (b)

Fig. 3. Reconstructions after sinogram is restored using (a) constant smoothing coefficients (Case 1) and (b) spatially varying vertical smoothing coefficient (Case 2).



(a) (b)

Fig. 4. (a) Horizontal and (b) vertical image profiles from Fig. 3a (dashed lines), Fig. 3b (solid lines), and Fig. 2a (original image) (dotted line).

V. Discussion

We have described an approach to object modeling in the sinogram domain which leads to a projection-space procedure for object reconstruction. It is based on knowledge of the convex support of the object and a prior estimate of the object's values within its convex support. The convex support was required to specify boundary conditions in the variational formulation while the prior estimate served to generate a sinogram which serves as a minimizer of a variational formulation devoid of measurements. Consistency of the sinogram is assured up to the first two moments of the Ludwig-Helgason conditions, although other methods are available to increase the order without requiring additional prior information. The simulation results are consistent with our expectation — that objects may be estimated with less boundary smoothing — but the effects of noise near the boundaries may overwhelm this benefit, requiring a compromise.

References

[1] A. Macovski, "Physical problems of computerized tomography," *Proc. IEEE*, vol. 71, no. 3, pp. 373–378, 1983.

[2] W. R. Madych and S. A. Nelson, "Reconstruction from restricted Radon transform data: resolution and ill-conditionedness," *SIAM J. Math. Anal.*, vol. 17, pp. 1447–1453, November 86.

[3] G. T. Herman, *Image Reconstruction from Projections.* New York: Academic Press, 1980.

[4] A. C. Kak and M. Slaney, *Principles of Computerized Tomographic Imaging.* New York: IEEE Press, 1988.

[5] J. L. Prince and A. S. Willsky, "Constrained sinogram restoration for limited-angle tomography," *Optical Engineering*, May 1990.

[6] J. L. Prince and A. S. Willsky, "A geometric projection-space reconstruction algorithm," *Linear Algebra and Its Applications*, vol. 130, pp. 151–191, 1990. Special Issue on Linear Algebra in Computed Tomography, ed. G. T. Herman.

[7] D. Ludwig, "The Radon transform on Euclidean space," *Comm. Pure Appl. Math.*, vol. 19, pp. 49–81, 1966.

[8] S. Helgason, *The Radon Transform.* Boston, MA: Birkhauser, 1980.

[9] F. B. Hildebrand, *Methods of Applied Mathematics.* Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1965.