TOMOGRAPHIC RECONSTRUCTION OF 3-D VECTOR FIELDS

Jerry L. Prince

Department of Electrical and Computer Engineering
Johns Hopkins University, Baltimore MD 21218

ABSTRACT

Inner product probe measurements are defined for tomographic reconstruction of 3-D vector fields. It is shown that one set of measurements is required to reconstruct an irrotational field, two are required to reconstruct a solenoidal field, and special probes are required to reconstruct the components of an arbitrary field.

1. INTRODUCTION

In recent years there has been a growing interest in tomographic reconstruction of vector fields. The primary driving force has been the realization that certain applications such as ultrasonic imaging, flow imaging, and ocean acoustic tomography have measurements that are inherently line integrals of the inner product of the vector field with a fixed unit vector. Norton [1] laid the groundwork for a theoretical treatment of this problem by showing that through a decomposition of the vector field into its irrotational and solenoidal components, conventional line integral projections — i.e., where the integral of the inner product of the field with a unit vector parallel to the line of integration is measured — can be used reconstruct the solenoidal component only. Braun and Hauck [2] extended this result by also considering measurements of the integral of the inner product of the field with a unit vector orthogonal to the line of integration. They showed that these data are sufficient to reconstruct the irrotational component of the 2-D vector field; thus, with two complete sets of measurements, each using a different unit vector for forming the inner product, the full vector field can be reconstructed. Norton has also shown that boundary measurements can be used to recover the irrotational component [3].

This paper generalizes this previous work by including a more general type of measurement, which we call *inner product probe measurements*, and by including three-dimensional vector fields. General conditions and formulas for reconstruction of an arbitrary vector field are given, as are the conditions and formulas for fields known to be either irrotational or solenoidal. We also develop formulas to extract the irrotational or solenoidal component of an arbitrary vector field using special inner product probe measurements. In order to use some standard results from the theory of the Radon transform (cf. [4]) we restrict the analysis to vector fields whose elements belong to either the space of rapidly decreasing C^∞ functions or the space of compactly supported C^∞ functions. Thus, some of the detailed issues associated with boundary conditions (cf. [2, 3]) are avoided.

The use of inner product measurements has several potential advantages in the imaging of vector fields. First, as pointed out by Braun and Hauck [2], forming the inner product has the effect of performing a derivative on the measured data, normally the first step in numerical inversion. Thus the measurement process itself performs an operation which is normally prone to numerical instability. Second, if a field is known to be either irrotational or solenoidal, then fewer measurements than are required to recover a general vector field may be used to recover just that component alone. This saves measurements and potentially reduces the effects of noise. Finally, some properties of the field, such as vorticity, can be calculated from a single component of the field [5]. Thus, if such a property is desired, it may be found from a smaller number of measurements than would be required to reconstruct the full vector field. We discuss these issues further in the conclusion.

2. PRELIMINARIES

A. The Radon Transform

Since we consider both 2-D and 3-D vector fields in this paper, we use a unified version of the inverse Radon transform formulas for functions defined on $\mathbb{R}^n$, where $n = 2$ or $n = 3$. Let f be a real scalar function defined on $\mathbb{R}^n$ belonging to either the class $\mathcal{L}$ of rapidly decreasing C^∞ functions (Schwarz space) or the class $\mathcal{D}$ of C^∞ functions with compact support. The n-D

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Radon transform of f can be written as [4]

$$\check{f}(t,\boldsymbol{\omega}) = \mathcal{R}f = \int_{\mathbb{R}^n} f(\mathbf{x})\delta(\mathbf{x}\cdot\boldsymbol{\omega} - t)d\mathbf{x}. \quad (1)$$

where $t \in \mathbb{R}^1$, $\boldsymbol{\omega} \in S^{n-1}$, and $\delta(\cdot)$ is the 1-D Dirac delta function. We define the Radon transform of a 3-D vector field $\mathbf{q} = \mathbf{q}(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$ as

$$\check{\mathbf{q}}(t,\boldsymbol{\omega}) = (\check{u}(t,\boldsymbol{\omega}), \check{v}(t,\boldsymbol{\omega}), \check{w}(t,\boldsymbol{\omega}))\,, \quad (2)$$

where we assume here and throughout this paper that the elements of $\mathbf{q}$ belong to either $\mathcal{L}$ or $\mathcal{D}$. The following properties of the Radon transform can be readily verified [see [4] for (a) and (b)]:

$$\mathcal{R}\{\frac{\partial f}{\partial x_i}\} = \omega_i \frac{\partial \check{f}}{\partial t}, \quad (3a)$$

$$\mathcal{R}\{\mathbf{p}\cdot\nabla f\} = \mathbf{p}\cdot\boldsymbol{\omega}\frac{\partial \check{f}}{\partial t}, \quad (3b)$$

$$\mathcal{R}\{\mathbf{p}\cdot(\nabla\times\mathbf{a})\} = (\mathbf{p}\times\boldsymbol{\omega})\cdot\frac{\partial \check{\mathbf{a}}}{\partial t}. \quad (3c)$$

Here, f and $\mathbf{a}$ are scalar and vector fields respectively and $\mathbf{p}$ is a vector which may depend on t and $\boldsymbol{\omega}$.

After a simple modification of a standard formula [6], the inverse Radon transform can be written as

$$f = \mathcal{R}^*\mathcal{K}'\mathcal{P}\check{f}. \quad (4)$$

where $\mathcal{R}^*$ is the adjoint operator defined by

$$(\mathcal{R}^*g)(\mathbf{x}) = \int_{|\boldsymbol{\omega}|=1} g(\mathbf{x}\cdot\boldsymbol{\omega},\boldsymbol{\omega})d\boldsymbol{\omega}\,,$$

and

$$\mathcal{K}' = \begin{cases} \frac{1}{2(2\pi i)^{n-1}}\left(\frac{\partial}{\partial t}\right)^{n-2} & \text{for } n \text{ odd}, \\ \frac{1}{2(2\pi i)^{n-1}} i\mathcal{H}\left(\frac{\partial}{\partial t}\right)^{n-2} & \text{for } n \text{ even}, \end{cases}$$

$$\mathcal{P} = \frac{\partial}{\partial t}.$$

Here $\mathcal{H}$ is the Hilbert transform operator. For notational convenience the subscript t will be used to denote partial derivative with respect to t — e.g., $\check{f}_t = \partial\check{f}/\partial t = \mathcal{P}\check{f}$.

B. Helmholtz's Theorem

According to Helmholtz's Theorem, a vector field $\mathbf{q}(x,y,z)$ can be uniquely written as [7]

$$\mathbf{q} = \mathbf{q}_I + \mathbf{q}_S\,, \quad (5a)$$

$$\mathbf{q}_I = \nabla\psi\,, \quad (5b)$$

$$\mathbf{q}_S = \nabla\times\mathbf{a}, \quad (5c)$$

where $\nabla\cdot\mathbf{a} = 0$. The scalar function ψ is called the *scalar potential* and the vector function $\mathbf{a}$ is called the *vector potential*. Since $\nabla\times\mathbf{q}_I = 0$, the component $\mathbf{q}_I$ is called *irrotational*; since $\nabla\cdot\mathbf{q}_S = 0$ the component $\mathbf{q}_S$ is called *solenoidal*. If $\mathbf{q}(x,y)$ is a 2-D vector field then Helmholtz's theorem states that $\mathbf{q}(x,y) = \mathbf{q}(x,y,0) = \nabla\psi(x,y) + \nabla\times\phi(x,y)\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is the unit vector pointing in the z-direction.

3. INNER PRODUCT MEASUREMENTS

We define the *inner product measurement* of the vector field $\mathbf{q}$ with respect to "probe" $\mathbf{p}$ as

$$g^{\mathbf{p}}(t,\boldsymbol{\omega}) = \int_{\mathbb{R}^n} \mathbf{p}\cdot\mathbf{q}(\mathbf{x})\,\delta(\mathbf{x}\cdot\boldsymbol{\omega} - t)d\mathbf{x}\,, \quad (6)$$

where the probe vector $\mathbf{p}$ may depend on t and $\boldsymbol{\omega}$ but not on $\mathbf{x}$. Now consider the measurements acquired from m probes $\mathbf{p}_1,\ldots,\mathbf{p}_m$. In matrix notation these measurements can be written as

$$\tilde{\mathbf{g}}(t,\boldsymbol{\omega}) = A(t,\boldsymbol{\omega})\check{\mathbf{q}}(t,\boldsymbol{\omega})\,, \quad (7)$$

where $\tilde{\mathbf{g}}(t,\boldsymbol{\omega}) = [g^{\mathbf{p}_1}(t,\boldsymbol{\omega})\ldots g^{\mathbf{p}_m}(t,\boldsymbol{\omega})]^{\mathrm{T}}$ and $A(t,\boldsymbol{\omega}) = [\mathbf{p}_1(t,\boldsymbol{\omega})\ldots\mathbf{p}_m(t,\boldsymbol{\omega})]^{\mathrm{T}}$.

If $m = n$ and A is invertible for all t and $\boldsymbol{\omega}$ then the complete Radon transform of the vector field $\mathbf{q}$ is easily recovered from the inner product measurements using

$$\check{\mathbf{q}}(t,\boldsymbol{\omega}) = A^{-1}(t,\boldsymbol{\omega})\tilde{\mathbf{g}}(t,\boldsymbol{\omega})\,, \quad (8)$$

and the vector field itself can be recovered using the inverse Radon transform as

$$\mathbf{q}(\mathbf{x}) = \mathcal{R}^{-1}\left\{A^{-1}(t,\boldsymbol{\omega})\tilde{\mathbf{g}}(t,\boldsymbol{\omega})\right\} = \mathcal{R}^*\mathcal{K}'\cap PA^{-1}\tilde{\mathbf{g}}. \quad (9)$$

If $m > n$ then the problem is overdetermined and one or more redundant equations can be removed until the reduced matrix is square and can be inverted. (Obviously, under noisy conditions one would seek some type of least squares estimate that uses all of the available data.) If $m < n$ then in general an arbitrary vector field cannot be recovered. The following section focuses on two special cases, however, in which the full field can be recovered from fewer than n inner product measurements. It should be noted that if $m = n$ and A is not invertible for all t and $\boldsymbol{\omega}$, full reconstruction may still be technically possible through limited data methods [8], but ill-posedness is a serious problem in these cases.

4. IRROTATIONAL AND SOLENOIDAL FIELDS

Many fields are restricted by physical laws to be either irrotational or solenoidal fields. For example, an electrostatic field generated by a collection of charged particles is irrotational, and a magnetic field is solenoidal. In this section we determine what inner product measurements are needed to allow perfect reconstruction when the field is known to be a particular one of these two types of fields.

A. Irrotational Fields

In this section the total field is assumed to be irrotational and is denoted by $\mathbf{q}_I$. Since $\mathbf{q}_I = \nabla\psi$ where ψ is a scalar field then from (3b)

$$g_I^{\mathbf{p}} = \mathcal{R}\{\mathbf{p}\cdot\mathbf{q}_I\} = \mathcal{R}\{\mathbf{p}\cdot\nabla\psi\} = \mathbf{p}\cdot\boldsymbol{\omega}\ \breve{\psi}_t\,. \quad (10)$$

Therefore, provided that $\mathbf{p}\cdot\boldsymbol{\omega}\neq 0$, $\breve{\psi}_t$ can be recovered from this measurement via

$$\breve{\psi}_t = \frac{g_I^{\mathbf{p}}}{\mathbf{p}\cdot\boldsymbol{\omega}}\,. \quad (11)$$

Then since $\breve{\psi}_t = \mathcal{P}\breve{\psi}$, the inverse Radon transform formula given in (4) combined with (11) yields

$$\mathbf{q}_I = \nabla\mathcal{R}^*\mathcal{K}'\frac{g_I^{\mathbf{p}}}{\mathbf{p}\cdot\boldsymbol{\omega}}\,, \qquad \mathbf{p}\cdot\boldsymbol{\omega}\neq 0\,. \quad (12)$$

The special case where $\mathbf{p}=\boldsymbol{\omega}$ is what Braun and Hauck [2] call the *transverse measurements*, and this leads to the especially attractive formula:

$$\mathbf{q}_I = \nabla\mathcal{R}^*\mathcal{K}' g_I^{\boldsymbol{\omega}}\,. \quad (13)$$

But (12) is a more general result which shows that any single inner product measurement $g_I^{\mathbf{p}}(t,\boldsymbol{\omega})$ can be used to reconstruct an irrotational field provided that the probe direction satisfies $\mathbf{p}\cdot\boldsymbol{\omega}\neq 0$ on $S^{n-1}\times\mathbb{R}^1$. Furthermore, both (12) and the special case (13) are valid in any dimension provided that $\mathbf{q}_I$ is the gradient of some scalar function ψ defined on $\mathbb{R}^n$.

B. Solenoidal Fields

In this section the total field is assumed to be solenoidal and is denoted by $\mathbf{q}_S$. Since $\mathbf{q}_S = \nabla\times\mathbf{a}$ it follows from (3c) that

$$g_S^{\mathbf{p}} = \mathcal{R}\{\mathbf{p}\cdot\mathbf{q}_S\} = \mathcal{R}\{\mathbf{p}\cdot(\nabla\times\mathbf{a})\} = (\mathbf{p}\times\boldsymbol{\omega})\cdot\breve{\mathbf{a}}_t\,. \quad (14)$$

Since $\mathbf{a}$ is solenoidal it follows that $\nabla\cdot\mathbf{a}=0$, and hence $\mathcal{R}\{\nabla\cdot\mathbf{a}\}=0$. But it can be easily shown using (3a) that $\mathcal{R}\{\nabla\cdot\mathbf{a}\} = \boldsymbol{\omega}\cdot\breve{\mathbf{a}}_t$, which implies $\boldsymbol{\omega}\cdot\breve{\mathbf{a}}_t = 0$. Therefore, $\breve{\mathbf{a}}_t$ is contained in the subspace orthogonal to $\boldsymbol{\omega}$; hence, in 3-D only two probe measurements are required to determine $\breve{\mathbf{a}}_t$. Denoting these probes by $\mathbf{p}_1$ and $\mathbf{p}_2$ we have

$$\tilde{\mathbf{g}}_S = B\breve{\mathbf{a}}_t\,, \quad (15)$$

where

$$\tilde{\mathbf{g}}_S = \begin{bmatrix} g_S^{\mathbf{p}_1} \\ g_S^{\mathbf{p}_2} \\ 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} (\mathbf{p}_1\times\boldsymbol{\omega})^{\mathrm{T}} \\ (\mathbf{p}_2\times\boldsymbol{\omega})^{\mathrm{T}} \\ \boldsymbol{\omega}^{\mathrm{T}} \end{bmatrix}.$$

Hence, the solenoidal field can be reconstructed using

$$\mathbf{q}_S = \nabla\times\mathcal{R}^*\mathcal{K}'B^{-1}\tilde{\mathbf{g}}_S\,. \quad (16)$$

It is easy to show that B is invertible — and therefore that complete measurements are available — if and only if $\mathbf{p}_1$, $\mathbf{p}_2$, and $\boldsymbol{\omega}$ are linearly independent.

A convenient pair of probes $\mathbf{p}_1$ and $\mathbf{p}_2$ can be defined by creating a particular orthonormal frame. Let $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ denote unit vectors in the x, y, and z directions respectively. Now let M be an orthogonal matrix with the properties that $\det M = +1$ and $\boldsymbol{\omega} = M\hat{\mathbf{k}}$. The two probes $\mathbf{p}_1 = \mathbf{p}_{\hat{\mathbf{j}}} \equiv M\hat{\mathbf{j}}$ and $\mathbf{p}_2 = \mathbf{p}_{\hat{\mathbf{i}}} \equiv -M\hat{\mathbf{i}}$ are orthogonal to $\boldsymbol{\omega}$ and to each other and clearly satisfy the above conditions. Furthermore, after evaluating the cross products it can be shown that $B = M^{\mathrm{T}}$, and since $M^{-1} = M^{\mathrm{T}}$ the full reconstruction formula is

$$\mathbf{q}_S = \nabla\times\mathcal{R}^*\mathcal{K}'M\tilde{\mathbf{g}}_S\,. \quad (17)$$

Only one probe is required to reconstruct a 2-D solenoidal field provided that the probe is not a multiple of $\boldsymbol{\omega}$. A particularly convenient choice for this probe is $\mathbf{p} = \boldsymbol{\xi} = (\sin\theta, -\cos\theta, 0)$, which is a unit vector in the x,y plane orthogonal to $\boldsymbol{\omega} = (\cos\theta, \sin\theta, 0)$. This special case gives the *longitudinal measurements* defined by Braun and Hauck [2]. In this case $\boldsymbol{\xi}\times\boldsymbol{\omega} = \hat{\mathbf{k}}$, which together with (14) implies that $g_S^{\boldsymbol{\xi}} = \hat{\mathbf{k}}\cdot\breve{\mathbf{a}}_t = \partial\breve{a}_z/\partial t$. Since Helmholtz's theorem for 2-D vector fields gives $\mathbf{a}$ with only a z-component, this measurement permits direct reconstruction of the vector potential without forming $\tilde{\mathbf{g}}_S$ and doing a matrix multiplication. Accordingly, the 2-D solenoidal field is reconstructed directly from $g_S^{\boldsymbol{\xi}}$ using

$$\mathbf{q}_S(x,y) = \nabla\times\mathcal{R}^*\mathcal{K}'g_S^{\boldsymbol{\xi}}(t,\boldsymbol{\omega})\hat{\mathbf{k}}\,. \quad (18)$$

C. Irrotational and Solenoidal Components

Braun and Hauck [2] discovered that the irrotational and solenoidal components of a 2-D vector field can be imaged separately using the transverse and longitudinal measurements, respectively. This result has a clear analogy in 3-D except that two measurements are now required to reconstruct the solenoidal component. This result was anticipated by Norton [1].

By linearity, the inner product measurement of an arbitrary vector field $\mathbf{q} = \mathbf{q}_S + \mathbf{q}_I$ is given by

$$g^{\mathbf{p}} = g_I^{\mathbf{p}} + g_S^{\mathbf{p}}\,. \quad (19)$$

It follows from (14) that $g_S^{\boldsymbol{\omega}} = 0$. Therefore, $g^{\boldsymbol{\omega}} = g_I^{\boldsymbol{\omega}}$ and $\mathbf{q}_I$ can be reconstructed using (12). Similarly, it follows from (10) that $g_I^{\mathbf{p_i}} = g_I^{\mathbf{p_j}} = 0$; hence $\mathbf{q}_S$ can be reconstructed using (17). Therefore, the irrotational or solenoidal component of an arbitrary 3-D vector field can be reconstructed separately using only one or two inner product measurement sets, respectively.

5. CONCLUSION

Previous efforts to develop the tomography of vector fields have focused on specific fixed vectors through which the inner product measurement is taken. We have generalized this concept to arbitrary inner product "probe" directions where these probes can depend on t and $\boldsymbol{\omega}$. We have shown very general conditions under which an arbitrary vector field can be reconstructed from probe measurements through a matrix inversion at each t and $\boldsymbol{\omega}$ followed by standard inverse Radon transform methods. We have also shown how fields known to be either irrotational or solenoidal can be reconstructed using fewer inner product measurements than is required for an arbitrary 3-D field. For these special cases, the probe directions are virtually unrestricted, which represents a much more general imaging scenario that has been previously reported in the 2-D case. Finally, we have shown how the irrotational or solenoidal component of an arbitrary vector field can be recovered separately using one or two inner product measurements, respectively.

Through generalization of the measurements to inner product probes and through extension of previous developments to three dimensions, we can now answer several questions that have been posed in the literature. For example, Norton [1] noted that standard line integral measurements can be used to recover the solenoidal component in 2-D vector fields, but not in 3-D fields. He suggested that one additional constraint — i.e., some a priori knowledge about the field — would be necessary to determine the second independent component that would arise in three dimensions. We have shown, however, that another measurement will suffice, using another linearly independent probe taken from within the plane of integration. Norton [1, 3] also noted that the irrotational component of a vector field can be recovered from measurements on the boundary. Braun and Hauck [2] showed in the 2-D case, however, that an inner product measurement using probe $\boldsymbol{\omega}$ (using our terminology) allows one to reconstruct the irrotational component. We have now generalized this result by showing that it is true in *any* dimension; thus, the reconstruction of an irrotational field requires only one inner product measurement, regardless of the dimension.

Several investigators have shown how to reconstruct various properties of 2-D vector fields such as the vorticity vector [5] and the index of refraction [9] using particular inner product measurements. Since these quantities can often be derived from the potentials rather than the full vector field it may be sufficient to reconstruct only ψ or $\mathbf{a}$, depending on what property is desired. Therefore, the requirement of taking a derivative — the divergence in the case of the irrotational component and the curl in the case of the solenoidal component — is eliminated [see Equation (12) and (16)]. Since the inner product measurement itself has the effect of taking the initial derivative itself, and since this last derivative is not required, there is potential for significant improvement in the signal to noise ratio of the reconstructed vector field property over other reconstruction methods.

6. REFERENCES

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