Adaptive Active Contour Algorithms for Extracting and Mapping Thick Curves

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Abstract

Thick curves arise naturally in certain applications such as magnetic resonance imaging of the brain; they can also arise in computer vision problems through morphological dilation of boundaries of objects. In this paper we describe two new adaptive active contour algorithms for the extraction and mapping of the skeleton of a thick curve. They are based on conditions that have been derived in previous work which guarantee uniqueness and fidelity of the solution. Both algorithms modify the regularization constant $K_0$ in an attempt to maintain convexity of the energy function while simultaneously improving the fidelity of the result. The first algorithm changes $K_0$ over time while the second adapts $K_0$ spatially. We evaluate both algorithms on experiments with synthetic curves; both demonstrate an improved performance compared to a fixed-parameter active contour algorithm.

1 Introduction

In this paper we address the problem of estimating the boundary of an object that is surrounded by a regular simple curve of constant thickness, which we call a thick curve (see Fig. 1). In particular, we seek the constant speed parametrization of the skeleton of this thick curve. Thick curves often appear in cross-sectional images of objects. For example, in magnetic resonance images of the brain, the cross-sections of the cortex resemble thick curves. Mapping these thick curves onto the unit interval is of interest in neuroscience, since it reveals information about the relative locations on the cortex of the functional centers of the brain. Moreover, since the dilation of a curve can be viewed as a thick curve — and any ordinary curve is a thick curve with zero thickness —, our work finds application to boundary estimation of more general classes of objects.

We present two algorithms which are based on an active contour model that was presented in [1]. Active contours have been developed by several investigators in recent years and have been used in several applications [2, 3, 4], including medical imaging [5], offering several advantages over methods such as edge linking or contour following. Their main drawback, however, is the lack of conditions under which they yield good solutions. In [1] we presented a new active contour model for thick curves and derived conditions under which it yields a good solution. The external forces of our model, which is briefly described in Section II, are based on a local averaging rather than differentiation, making the proposed method more robust to noise and discretization errors.

We study two approaches having a variable regularization constant. The first is a time adaptive formulation that dramatically improves the performance of the active contour algorithm, at the expense, however, of increased computational requirements. The second is a spatially adaptive approach which is a good compromise between the fixed parameter and the time adaptive formulations. We apply both of these adaptive algorithms to synthetic data and demonstrated a better performance in comparison to the original formulation.

2 Active Contour Algorithm

In this section we give an overview of the active contour algorithm presented in [1]. Let $\alpha(s) : [0, 1] \to \mathbb{R}^2$ be a constant velocity plane curve with unit normal vector $N(s)$, as shown in Fig. 1. A thick curve $C$ is defined using two boundary curves given by $\alpha_1(s) = \alpha(s) + wN(s)/2$ and $\alpha_2(s) = \alpha(s) - wN(s)/2$, where $w$ is a width parameter satisfying $w > 0$. Our goal is to find $\alpha(s)$ from the indicator function of $C$, which
we call the mass function:
\[ m(x) = m(x, y) = \begin{cases} 1, & x \in C, \\ 0, & \text{otherwise}. \end{cases} \] (1)

We also define a center of mass function \( c(x, y) \) for each image point as the center of the mass of the thick curve included in a circular neighborhood centered around it. The active contour algorithm presented in [1] is based on a variational formulation in which the solution is the vector function \( \tilde{z}(s), s \in [0, 1], \) that minimises
\[ E = E_F + E_B, \] (2)
where
\[ E_F = \int_0^1 \| x(s) - c(x(s)) \|^2 ds \] (3)
\[ E_B = K_0 \int_0^1 \left( \left( \frac{dz(s)}{ds} \right)^2 + \left( \frac{dy(s)}{ds} \right)^2 \right) ds, \] (4)
subject to the boundary conditions:
\[ z(0) = \alpha, \quad z(1) = \beta, \quad y(0) = \gamma, \quad y(1) = \delta. \] (5)
The parameter \( K_0 \) in (4) determines the elasticity of the active contour and is of major importance in this paper.

In order to solve this variational problem numerically, we discretize the functions \( x(s) \) and \( y(s) \), by sampling them with \( N + 1 \) points each, forming the vector \( d \). Similarly, we discretize the function \( c(s) \), forming the vector \( f \). The energy function \( E \) in discrete form becomes
\[ \tilde{E}(d) = E_F(d) + E_B(d), \] (6)
where
\[ E_F(d) = \frac{1}{N} (d - f)^T (d - f), \] (7)
\[ E_B(d) = K_0 N \left[ d^T A d - 2 b^T d \right] + C, \] (8)
where \( b \) is a vector associated with the boundary conditions, \( C \) is a constant, and
\[ A = \text{diag}(B), \] (9)
\[ B = \text{diag}\{-1, 2, -1\}. \] (10)

If \( d \) minimises \( \tilde{E}(d) \) then it must also satisfy
\[ \frac{N}{2} \nabla E_F(d) + K_0 N^2 A d = K_0 N^2 b. \] (11)

This discrete formulation will be referred to as (DVP1).

Equation (11) is only a necessary condition for optimality; hence, it is satisfied by both local and global minima. Therefore, any of these minima can be obtained as the solution if a deterministic descent optimisation method is employed. Local minima can be eliminated if the regularisation constant \( K_0 \) is chosen sufficiently large. In fact, in [1] we derived a lower bound \( K_{LB} \) such that if \( K_0 \) is larger than \( K_{LB} \) the energy function is convex, hence, (11) is satisfied at only one point, the global minimum. In the following section we show that \( K_{LB} \) is a sharp bound.

3 A Sharp Lower Bound on \( K_0 \)

The lower bound \( K_{LB} \) derived in [1] assumes that small pieces of the thick curve can be approximated by thick arcs. For each point \( x_i \) of the active contour, we defined a polar coordinate system with its origin being the center of the arc. In this coordinate system, we denoted the radial and angular coordinates of \( x_i \) by \( r_i \) and \( \phi_i \) respectively. We also defined the function \( n_i \), as the distance between \( (x_i, y_i) \) and the center of mass \( (\bar{u}, \bar{v}) = (u(x_i, y_i), v(x_i, y_i)) \) and \( n'_i \) and \( n''_i \) as its first and second derivatives with respect to \( r_i \). We then showed that \( \tilde{E} \) is guaranteed to be convex if \( K_0 > K_{LB} \), where
\[ K_{LB} = \frac{-1}{N^2 \lambda_{min}} \begin{bmatrix} n_i n''_i + (n'_i)^2 \\ n_i n'_i \\ r_i \end{bmatrix}. \] (12)

Here, \( \lambda_{min} \) is the minimum eigenvalue of \( A \). In the claim that follows we prove that \( K_0 > K_{LB} \) is also a necessary condition.

Claim 1: \( \tilde{E} \) is convex if and only if \( K_0 > K_{LB} \).

Proof: In [1] we show that \( K_0 > K_{LB} \) is a sufficient condition for convexity of the energy function. To prove that it is also necessary, we will show that \( K_{LB} \) is a sharp lower bound and, therefore, if \( K_0 \) is chosen to be less than \( K_{LB} \), there exists a solution around which the energy function is not convex. We will do this by constructing such a solution.

Let \( H_F, H_P, \) and \( H_B \) be the Hessian matrices of the energy functions \( E, E_F, \) and \( E_B \) respectively. It is easy to see from (2) that
\[ H = H_F + H_B. \] (13)

It can be shown (see [1]) that \( H_F = [f_{ij}] \), where
\[ f_{ii} = 2 \frac{1}{N} \begin{bmatrix} a_i \cos^2 \phi_i + b_i \sin^2 \phi_i, & i < N \\ a_i \sin^2 \phi_i + b_i \cos^2 \phi_i, & i = N \end{bmatrix}, \] (14)
\[ f_{ij} = \frac{2}{N} \begin{bmatrix} (a_i - b_i) \cos \phi_i \sin \phi_i, & 0, \end{bmatrix} \] (15)
with

\[ a_i = (n_i n_i' + (n_i')^2), \quad b_i = \frac{n_i n_i'}{r_i} \] (16)

and that \( H_B = 2K_e N A \). Now consider a very special active contour configuration for which

\[ \phi_i = k r, \quad k \in \{0, 1\}, \] (17)

\[ \frac{2}{N} a_i = \lambda_1, \quad \frac{2}{N} b_i = \lambda_2, \quad i = 1, \ldots, N - 1, \] (18)

where \( \lambda_1, \lambda_2 \) are arbitrary constants. We will now show that for this specific case

\[ \lambda_k \{ H \} = \lambda_k \{ H_F \} + \lambda_k \{ H_E \}, \quad k = 1, 2, \ldots, 2N - 2, \] (19)

where \( \lambda_k \{ \} \) represents the \( k \)-th eigenvalue of the matrix argument. Given (17) and (18) and using (14) and (15) we see that

\[ H_F = \text{block-diag}(\lambda_1 I, \lambda_2 I) \] (20)

It can also be shown that the diagonalization of \( H_B \) takes the form

\[ H_B = 2K_e N V^T \begin{bmatrix} \Lambda_B & 0 \\ 0 & \Lambda_B \end{bmatrix} V, \] (21)

where \( \Lambda_B \) is a diagonal matrix with its diagonal elements equal to the eigenvalues of \( A \), which were explicitly derived in [1], and \( V \) is a block-diagonal matrix with each block being an orthonormal matrix \( U \). Clearly

\[ V^T \begin{bmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{bmatrix} V = \begin{bmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{bmatrix}; \] (22)

therefore

\[ H = V^T \begin{bmatrix} 2K_e N \Lambda_B + \lambda_1 I & 0 \\ 0 & 2K_e N \Lambda_B + \lambda_2 I \end{bmatrix} V, \] (23)

which implies (19).

For this specific case (12) becomes

\[ K_{LB} = \left( -\frac{N}{2} \lambda_3 \right)/(N^2 \lambda_{\min}) = -(\lambda_2)/(2N \lambda_{\min}). \] (24)

Now let \( d = [0, u_{\min}]^T \) be the eigenvector of \( A \) corresponding to \( \lambda_{\min} \). Using (23) and the orthogonality of \( U \) we can show that

\[ d^T H d = 2K_e N \lambda_{\min} + \lambda_2. \] (25)

Then using (24) and (25), we see that setting \( K_e < K_{LB} \) implies

\[ d^T H d < 2 \left( -\frac{\lambda_2}{2N \lambda_{\min}} \right) N \lambda_{\min} + \lambda_2 = 0. \] (26)

This equation implies that \( H \) is not positive definite, which in turn implies that \( \phi \) is not convex. Therefore, \( K_{LB} \) is a sharp lower bound.

Although the condition \( K_e > K_{UB} \) implies the convexity of the energy function and, hence, the uniqueness of the solution to (11), this solution is not necessarily a good solution. In fact, if \( K_e \) is chosen too large, this unique solution can be an oversmoothed version of the desired curve \( \phi(s) \). In [1] we showed that for a good reconstruction of \( \phi(s) \), \( K_e \) must not exceed an upper bound \( K_{UB} \), where \( K_{UB} \) is determined using a frequency analysis of the thick curve.

In summary, if

\[ K_{LB} < K_e < K_{UB}, \] (27)

a good solution, which is also unique, is guaranteed. In practice it is rarely possible to satisfy (27), especially if the thick curve is convoluted, in which case \( K_{LB} \) often is larger than \( K_{UB} \). This situation leads to final active contour configurations that are either oversmoothed versions of \( \phi(s) \) or correspond to local minima.

In the following two sections we introduce a new active contour model, based on the notion of a variable regularization coefficient, which will turn out to significantly improve the ability of the active contour to avoid local minima, balancing at a configuration close to \( \phi(s) \).

4 Time Adaptive Algorithm

In this section we propose an active contour model with a time varying regularization coefficient, which we denote by \( K_e(t) \). In this model, \( K_e(t) \) is initialized with a large value, which guarantees convexity of the energy function. As the active contour approaches the thick curve \( K_e(t) \) decreases, allowing the active contour to grasp the finer details of the thick curve without being trapped in local minima. This is sensible since, as we will see, \( K_{LB} \) approaches 0 as the active contour approaches \( \phi(s) \).

Under the same assumptions that led to (12) we can derive explicit formulas for \( n'(r) \) and its derivatives \( n'(r) \) and \( n''(r) \) (see [1]). In Fig. 2 we plot the quantities \( n(r)n''(r) + \left( n'(r) \right)^2 \) and \( n(r)n'(r) \) as functions of \( r \), for different ratios of the radius of curvature \( R \) of the thick curve over the neighborhood radius \( \rho \). From these plots and from (12) we can deduce that

\[ \lim_{n_i \to 0, \forall i} K_{LB} = 0. \] (28)

This turns out to be a very interesting property of \( K_{LB} \), since it implies that if we assume that the active
contour is sufficiently close to the central layer of the thick curve, we can always find a value for \( K_a \) that will satisfy (27) and therefore yield a good solution. It is a common assumption in the active contour literature that the active contour is initialized sufficiently close to the curve to be reconstructed. Although this assumption may be reasonable in many applications, it is very restrictive for many cases where human intervention is to be kept minimal. We propose a new algorithm, which employs a time varying regularization constant, starting from a large value and decreasing in time.

We vary \( K_a(t) \) according to the following rule

\[
K_a(t) = K_a + g(t)
\]

where \( g(t) \) is a positive decreasing function that approaches 0 as \( t \) increases. The variable \( t \) denotes time, which is proportional to the number of iterations. If we choose \( g(0) \) sufficiently large, the energy function will be convex and our deterministic optimization method will avoid the local minima. As the active contour approaches the central layer of \( C \), \( g(t) \) is allowed to decrease. However, if we assume that \( g(t) \) decreases at the same rate as \( K_{LB} \), the convexity of the energy function is preserved. If the active contour is sufficiently close to this central layer and \( K_a \) is small enough then both the lower and the upper bound are satisfied and a good solution is obtained.

In our initial experiments we chose the following linear form of \( g(t) \):

\[
g(t) = \begin{cases} 
g(0) \frac{t}{T}, & t \leq T \\
0, & t > T 
\end{cases}
\]

where \( g(0)/T \) determines the rate of decrease of \( g(t) \). Although this might not necessarily be the best choice for the function \( g(t) \), it proved to dramatically increase the range of \( K_a \) that yields good solutions. To demonstrate the performance of this time adaptive algorithm, in Fig. 3 we plot the maximum distance of the reconstructed curve from the image of \( \alpha(s) \), for different ratios of \( \frac{K_a(0)}{K_o} \). In Fig. 3, all of the error curves correspond to experiments where we used \( t_{final} > T \). Therefore, in all of the experiments our algorithm terminated with \( K_a(t) = K_o \), although the initial values of \( K_a \) were different. We can clearly see from this figure that the critical value of \( K_a \), below which the solution corresponds to a "bad" local minimum, steadily decreases as higher values of \( \frac{K_a(0)}{K_o} \) are adopted. Especially in the case of \( \frac{K_a(0)}{K_o} = 10,000 \), the error plot is flat even for \( K_a \)-values two orders of magnitude lower than the critical \( K_a \)-value of the fixed \( K_a \) algorithm. From Fig. 3 it becomes clear that the time adaptive algorithm alleviates the problem of choosing the regularization constant, since for a broad range of \( K_a \)-values we obtain a good solution.

The final active contour configurations for two different values of the ratio \( \frac{K_a(0)}{K_o} \), which are shown in Fig. 4, also reveal the improvement in performance obtained using time adaptivity. The contour of Fig. 4a is the solution to the fixed-\( K_a \) variational problem, while the one of Fig. 4b is the solution to the time adaptive formulation, with \( \frac{K_a(0)}{K_o} = 10,000 \). In both cases, \( K_o \) was chosen to be equal to \( 2 \times 10^{-9} \). The contour of Fig. 4a however, contains two major splits, an indication that it corresponds to a local minimum of \( \hat{E} \). These splits are not present in Fig. 4b.

One can make a more precise statement about the choice of \( g(t) \). From the convexity condition, we can see that if we choose

\[
g(t) > K_{LB}(t) - K_a,
\]

the convexity of \( \hat{E} \) is always guaranteed. In order to determine \( K_{LB}(t) \) the knowledge of the quantities \( n_i \), \( n'_i \), and \( r_i \) is required. The first is given, since the position of the center of mass is defined for every point. The second can be approximated by a finite difference. The third can be also estimated at each point, under the assumption that, locally, the thick curve can be approximated by a thick arc. Although this approach for varying \( K_a \) is promising, since it guarantees convexity of \( \hat{E} \) at any time, we do not have any experimental results at this time.

5 Spatially Adaptive Algorithm

Although the performance of the time-adaptive algorithm is impressive, its computations are time consuming since \( K_a \) sweeps a very large interval before it eventually assumes its final value. In this section, we propose an alternative approach based on a spatially varying \( K_a \), denoted by \( K_a(s) \), which is a compromise between the inadequacy of the fixed \( K_a \) algorithm to avoid local minima and the increased computational load of the time varying \( K_a \).

Since local minima appear as splits in the active contour, it is reasonable to increase \( K_a \) at the points where these splits occur, while keeping \( K_a \) small over the remainder of the curve. In this way, one hopes to avoid local minima and maintain fast convergence. In
our formulation we vary $\bar{K}_\alpha(s)$ as follows:

$$
\bar{K}_\alpha(s) = K_\alpha \left( 1 + \theta U \left( |v(s)| - \frac{L}{N} \right) \right) \tag{32}
$$

where $U(\cdot)$ is the step function, $\theta$ is the adaptivity coefficient, and $L$ and $v(s)$ are the length and velocity of $\alpha(s)$ respectively. In contrast to the algorithm presented in the previous section, this choice of $\bar{K}_\alpha(s)$ does not necessarily imply the convexity of the energy function. It does imply however, as is shown in the following proposition, that it can only increase the convexity of $E$.

Claim 2: Let $H'_B$ be the Hessian matrix of $\tilde{E}_B$ with $\bar{K}_\alpha(s)$ replacing $K_\alpha$. Then

$$
\lambda_{\min}(H'_B) \geq \lambda_{\min}(H_B). \tag{33}
$$

\textbf{Proof:} The elasticity term of the spatially adaptive $K_\alpha$ formulation is, in discrete form,

$$
\tilde{E}_B(d) = N \sum_{t=0}^{N-1} K_t \left[ (\eta_{t+1} - \eta_t)^2 + (\eta_{t+1} - \eta_t)^2 \right]. \tag{34}
$$

From this equation it is not hard to see that

$$
H'_B = H_B + NE, \tag{35}
$$

where $E$ is a block diagonal matrix, each block being the matrix

$$
E_\delta = \text{tridiag}\{-2\delta_{\epsilon,1}, 2\delta_{\epsilon,1} - 2\delta_{\epsilon,1}, 2\delta_{\epsilon,1} - 2\delta_{\epsilon,1}\} \tag{36}
$$

with

$$
\delta_{\epsilon,1} = K_t - K_\alpha > 0.
$$

Therefore, the eigenvalues of $E$, which lie in the union of the disks centered on $2\epsilon_{i,1}$ and have radii $2\epsilon_{i,1}$, are non-negative. Since

$$
\lambda_{\min}(H'_B) \geq \lambda_{\min}(H_B) + N\lambda_{\min}(E), \tag{33}
$$

(33) follows immediately. \hfill \Box

Although we have shown that the convexity of the energy function cannot decrease if we introduce positive spatial variation to $K_\alpha(s)$, a precise study of the way it is affected has not been undertaken yet.

We have applied this algorithm to the same set of synthetic data and its performance proved to be better than the fixed-$K_\alpha$ algorithm, while its rate of convergence better than the time adaptive algorithm. Therefore, it constitutes a good compromise between the previous two algorithms. In Fig. 5 we plot the maximum distance error measure as a function of $K_\alpha$ for various values of $\theta$. It is clear from this plot that the critical value of $K_\alpha$ decreases as the adaptivity coefficient increases. This implies that the range of $K_\alpha$ values which yield a good reconstruction increases. In Fig. 6 we show the final active contour configuration for $\theta = 0$ and $\theta = 4$. The curve of Fig. 6a contains two splits, one on the left and one on the right. These splits are not present in Fig. 6db, where a spatially varying $K_\alpha$ was employed.

6 Summary

We have presented two new adaptive active contour algorithms that reconstruct the nearly constant speed parametrization of the skeleton of a thick curve. They are based on a time and spatially varying regularization constant respectively. The time adaptive algorithm was shown to dramatically improve the performance of the active contour model at the expense, however, of increased computational requirements. The spatially adaptive algorithm was shown to be a good compromise between the non-adaptive and the time-adaptive algorithms. Although the way that the regularization parameter $K_\alpha$ is varied in these two formulations is somewhat heuristic, the results are promising. A more rigorous formulation for varying $K_\alpha$ based on the bounds that we have derived is one of the topics of our future research.

References


Fig. 1: The thick curve geometry.

Fig. 2: Plots of $n(r)^2$ and $(n'(r))^2$ and $\frac{n(r)n'(r)}{r}$ (see text).

Fig. 3: Error plots for the time adaptive algorithm.

Fig. 4: The final active contour configurations for the non-adaptive (left) and the adaptive (right) algorithm.

Fig. 5: Error plots for the spatially adaptive algorithm.

Fig. 6: The final active contour configurations for the non-adaptive (left) and the spatially adaptive (right) algorithm, for the same $K_a$.