ON DIV-CURL REGULARIZATION FOR MOTION ESTIMATION IN 3-D VOLUMETRIC IMAGING

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ABSTRACT

We consider the classical optical flow algorithm due to Horn and Schunck for estimating the motion of brightness patterns between image pairs. We use a modified smoothness condition based on the divergence and curl of the velocity field. In previous work, we have developed well-posed stochastic state-space models for these optical flow methods in two dimensions. This paper extends our results to 3-D. We first show that by using the first order DIV-CURL spline, it is not possible to obtain a first order linear differential well-posed model in 3-D. Next, we employ the second order DIV-CURL spline smoothness condition and develop well-posed state-space models.

I. Introduction

Optical Flow (OF) is the apparent motion of brightness patterns in an image sequence. In [1], Horn and Schunck proposed a differential method to compute the optical flow from pairs of images. A brightness constancy assumption was made — that the brightness of a material point does not change with time — which, combined with regularization, was used in a variational framework to solve for the velocity components. In our research, we have focused attention on developing state-space models for such variational problems.

In [2], Rougee, Levy, and Willsky formulated Horn and Schunck's OF approach as a linear estimation problem, which yielded a solution identical to that of Horn and Schunck. The advantages of such a formulation are that the underlying motion model is made explicit and that a priori measures of error can be computed. However, we demonstrated in [3] that Rougee's underlying state-space model is over-determined, which makes it unrealizable, and therefore non-physical. We also showed that by using smoothness conditions on the first order differential invariants of the vector field, namely the divergence and the curl [4], we can obtain a well-posed stochastic model, which also leads to Horn and Schunck's solution.

Our overall objective is to use these motion estimation techniques for accurately quantifying the myocardial motion in the left ventricle in normal and ischemic hearts (see [5], [6]). Since the previous work was set in two dimensions, and cardiac motion estimation is inherently 3-D, it is our goal in this paper to extend our previous results to 3-D. In the following section, we consider two state-space models for 3-D OF using divergence and curl smoothness conditions. We first show that using the *first order* DIV-CURL spline, we cannot get an equivalent first order well-posed linear stochastic state space model in 3-D. However, overdetermined models may be readily obtained for this case, and one such model is shown below. We then show that by using the *second order* DIV-CURL spline, we can obtain a well-posed state space model.

II. Background

Let (ψ_x, ψ_y, ψ_z) denote the gradient of the image intensity ψ, and let ψ_t denote the spatial time derivative of image intensity. In two-frame OF algorithms, ψ (a function of spatial co-ordinates) is known at two time instants, and ψ_x, ψ_y, ψ_z, and ψ_t are numerically estimated from these images (as was done in [1] in the 2-D case). The objective is to determine the velocity field $\mathbf{v} = (u, v, w)$ which accounts for the brightness changes due to motion between the two images.

Horn and Schunck's algorithm assumes that the brightness of material points does not change as they move. This condition yields the brightness constraint equation which states that the material time derivative of brightness $\dot{\psi}$ is zero :

$$\dot{\psi} = \nabla\psi \bullet \mathbf{v} + \psi_t = 0 . \tag{1}$$

Since (1) alone is not sufficient to solve for $\mathbf{v}$, Horn and Schunck regularized the problem by imposing smooth-

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ness conditions based on penalizing the squared magnitude of the gradients of the velocity components.

In our work, we use Suter's DIV-CURL OF approaches [7], where smoothness is imposed on the divergence and curl of $\mathbf{v}$. This approach is attractive since it provides an easy way of incorporating knowledge about physical properties of the flow field. For example, since heart wall motion is known to be approximately incompressible in 3-D, the DIV-CURL splines are better suited to model this behavior.

Following Suter, we consider the problem of finding $\mathbf{v}$ that minimizes one of the following :
the *first order DIV-CURL spline*

$$\int \{\dot{\psi}^2 + \alpha \parallel \operatorname{div} \mathbf{v}\|^2 + \beta \parallel \operatorname{curl} \mathbf{v}\|^2\} dxdydz\,, \qquad (2)$$

or the *second order DIV-CURL spline*

$$\int \{\dot{\psi}^2 + \alpha \parallel \nabla \operatorname{div} \mathbf{v}\|^2 + \beta \parallel \nabla \operatorname{curl} \mathbf{v}\|^2\} dxdydz\,, \qquad (3)$$

where $\nabla \operatorname{curl} \mathbf{v}$ stands for the sum of the gradients of components of $\operatorname{curl} \mathbf{v}$. Eq. (2) above is equivalent to the standard Horn and Schunck functional when $\alpha = \beta$ (see also [4], [8]).

The solution of these variational problems is given by the Euler-Lagrange equations. The solution of (2) can be shown to be

$$\begin{aligned} D_0(x,y,z)u + D_1(x,y)v + D_1(x,z)w &= \psi_x \dot{\psi} \\ D_0(y,z,x)v + D_1(y,z)w + D_1(y,x)u &= \psi_y \dot{\psi} \quad (4) \\ D_0(z,x,y)w + D_1(z,x)u + D_1(z,y)v &= \psi_z \dot{\psi} \end{aligned}$$

where

$$\begin{aligned} D_0(x,y,z) &= \alpha \frac{\partial^2}{\partial x^2} + \beta \frac{\partial^2}{\partial y^2} + \beta \frac{\partial^2}{\partial z^2}\,, \\ D_1(x,y) &= (\alpha - \beta) \frac{\partial^2}{\partial x \partial y}\,. \end{aligned}$$

Similarly, the solution of (3) is given by (5) on the next page.

III. Stochastic Models

Our goal is to develop an alternative linear smoothing theory viewpoint for the above variational problems. For this purpose, let us consider state-space models of the following form [9]:

$$L\mathbf{v}(\mathbf{r},t) = B\mathbf{u}(\mathbf{r},t) \qquad \text{for } \mathbf{r} \in \Omega \qquad (6a)$$

$$F\mathbf{v}_b(s) = \mathbf{u}_b(s) \qquad \text{for } s \in \partial\Omega \qquad (6b)$$

where the velocity is assumed to be defined on a regular domain Ω with boundary $\partial\Omega$. Here $\mathbf{u} \sim N(0, \sigma_u^2 I)$, $\mathbf{u}_b(s) \sim N(0, \sigma_u^2 I)$, and the subscript b implies a restriction of the variable to the boundary $\partial\Omega$, and the spatial coordinate $\mathbf{r} = (x, y, z)$. The observation equations are

$$\mathbf{y} = C\mathbf{v} + \mathbf{w} \qquad \text{on } \Omega \qquad (7a)$$

$$\mathbf{y}_b = C\mathbf{v}_b + \mathbf{w}_b \qquad \text{on } \partial\Omega \qquad (7b)$$

where $\mathbf{w} \sim N(0, \sigma_w^2 I)$ and $\mathbf{w}_b \sim N(0, \sigma_w^2 I)$. Note that the dependence on $\mathbf{r}$ and t is suppressed in the above.

The solution to this linear smoothing problem is found by using the theory of complimentary processes, and Adams et al. [9] have shown that the linear minimum mean squared error estimate $\hat{\mathbf{v}}$ is given by

$$\begin{bmatrix} L & -\sigma_u{}^2 BB^* \\ \frac{1}{\sigma_w{}^2} C^* C & L^\dagger \end{bmatrix} \begin{bmatrix} \hat{\mathbf{v}} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sigma_w{}^2} C^* y \end{bmatrix} \qquad (8)$$

on Ω, where $L^\dagger$ is the adjoint of L, and $\hat{\lambda}$ is the complimentary process of $\mathbf{v}$. A similar equation exists for the estimate of $\mathbf{v}_b$ and is not shown here. $L^\dagger$ is defined by the Green's identity

$$\langle Lx, \lambda \rangle_{L_2} = \langle x, L^\dagger \lambda \rangle_{L_2} + \langle x_b, E\lambda_b \rangle_{H_b} \qquad (9)$$

where L_2 and H_b are Hilbert spaces of vector square integrable functions defined on Ω and $\partial\Omega$ respectively.

Our objective is to specify the parameters of (6) and (7) so that the solution (8) is identical to either (4) or (5). To match the brightness constraint equation, we set the output gain C to be the row vector $[\psi_x \quad \psi_y \quad \psi_z]$, and the measurements $\mathbf{y} = -\psi_t$ and $\mathbf{y}_b = -\psi_t$. We need now simply to pick the appropriate L and B so that the solution in (8) matches either (4) or (5), and to pick F to match the boundary conditions.

A. First order DIV-CURL spline

In this section, we develop stochastic models for (2). It can be shown that if we let $B = I_4$ and choose

$$L = \begin{bmatrix} \sqrt{\alpha}\frac{\partial}{\partial x} & \sqrt{\alpha}\frac{\partial}{\partial y} & \sqrt{\alpha}\frac{\partial}{\partial z} \\ 0 & -\sqrt{\beta}\frac{\partial}{\partial z} & \sqrt{\beta}\frac{\partial}{\partial y} \\ \sqrt{\beta}\frac{\partial}{\partial z} & 0 & -\sqrt{\beta}\frac{\partial}{\partial x} \\ -\sqrt{\beta}\frac{\partial}{\partial y} & \sqrt{\beta}\frac{\partial}{\partial x} & 0 \end{bmatrix}, \qquad (10)$$

then the LMMSE solution in (8) is identical to (4). In this sense, this gives an equivalent state space model for (2) and is the first result of this paper. However, we note that this model sets the divergence of $\mathbf{v}$ and the three components of the curl of $\mathbf{v}$ to four independent white noise processes. It is not possible to solve

$$
\begin{aligned}
&\alpha\frac{\partial^4 u}{\partial x^4}+\beta\frac{\partial^4 u}{\partial y^4}+\beta\frac{\partial^4 u}{\partial z^4}+(\alpha+\beta)\frac{\partial^4 u}{\partial x^2\partial y^2}+(\alpha+\beta)\frac{\partial^4 u}{\partial x^2\partial z^2}+2\beta\frac{\partial^4 u}{\partial y^2\partial z^2}+\\
&\quad(\alpha-\beta)\left[\frac{\partial^4 v}{\partial x^3\partial y}+\frac{\partial^4 v}{\partial x\partial y^3}+\frac{\partial^4 v}{\partial x\partial y\partial z^2}+\frac{\partial^4 w}{\partial x^3\partial z}+\frac{\partial^4 w}{\partial x\partial z^3}+\frac{\partial^4 w}{\partial x\partial y^2\partial z}\right] = \psi_x\dot{\psi}\\
&\alpha\frac{\partial^4 v}{\partial y^4}+\beta\frac{\partial^4 v}{\partial z^4}+\beta\frac{\partial^4 v}{\partial x^4}+(\alpha+\beta)\frac{\partial^4 v}{\partial y^2\partial z^2}+(\alpha+\beta)\frac{\partial^4 v}{\partial y^2\partial x^2}+2\beta\frac{\partial^4 v}{\partial z^2\partial x^2}+\\
&\quad(\alpha-\beta)\left[\frac{\partial^4 w}{\partial y^3\partial z}+\frac{\partial^4 w}{\partial y\partial z^3}+\frac{\partial^4 w}{\partial y\partial z\partial x^2}+\frac{\partial^4 u}{\partial y^3\partial x}+\frac{\partial^4 u}{\partial y\partial x^3}+\frac{\partial^4 u}{\partial y\partial z^2\partial x}\right] = \psi_y\dot{\psi} \qquad (5)\\
&\alpha\frac{\partial^4 w}{\partial z^4}+\beta\frac{\partial^4 w}{\partial x^4}+\beta\frac{\partial^4 w}{\partial y^4}+(\alpha+\beta)\frac{\partial^4 w}{\partial z^2\partial x^2}+(\alpha+\beta)\frac{\partial^4 w}{\partial z^2\partial y^2}+2\beta\frac{\partial^4 w}{\partial x^2\partial y^2}+\\
&\quad(\alpha-\beta)\left[\frac{\partial^4 u}{\partial z^3\partial x}+\frac{\partial^4 u}{\partial z\partial x^3}+\frac{\partial^4 u}{\partial z\partial x\partial y^2}+\frac{\partial^4 v}{\partial z^3\partial y}+\frac{\partial^4 v}{\partial z\partial y^3}+\frac{\partial^4 v}{\partial z\partial x^2\partial y}\right] = \psi_z\dot{\psi}
\end{aligned}
$$

for $\mathbf{v}$ given this input; therefore, this model is overdetermined and non-physical. In 2-D, the above strategy led to a well-posed model since in 2-D the divergence and curl can be specified by one scalar field each.

It is natural to ask whether a change of coordinates or some other choice of L and B could lead to a well-posed model in 3-D having the same solution as (4). In our second result of this paper, we have shown that there is no such well-posed model having the form of (6). The proof of this result is sketched below for a 3×3 operator L and $B = I_3$. The result can be generalized for any models of the form (6).

For a well-posed model, we want to find a 3×3 linear differential operator such that (8) matches (4). This requires finding L such that

$$
L^\dagger L = \begin{bmatrix} D_0(x,y,z) & D_1(x,y) & D_1(x,z) \\ D_1(y,x) & D_0(y,z,x) & D_1(y,z) \\ D_1(z,x) & D_1(z,y) & D_0(z,x,y) \end{bmatrix}. \qquad (11)
$$

We note that the determinant of this matrix operator in symbolic form must satisfy

$$
\det\left[L^\dagger L\right] = [\det L]^2 \qquad (12)
$$

However, the determinant of $L^\dagger L$ evaluated from (11) does not have a square-root and is irreducible. This shows that we cannot factor the above operator into L and $L^\dagger$ as required.

So, we conclude that for the first-order DIV-CURL spline, although the variational solution is well-defined, an equivalent well-posed stochastic interpretation, if it exists at all, is not of the form (6).

B. Second order DIV-CURL spline

We now consider the variational problem (3). The Euler-Lagrange equations for this case consist of a system of three coupled fourth order PDE's in $u, v,$ and w, as shown in (5). It can be verified that choosing $B = I_3$ and

$$
L = \begin{bmatrix} D_2(x,y,z) & D_3(x,y) & D_3(x,z) \\ D_3(y,x) & D_2(y,z,x) & D_3(y,z) \\ D_3(z,x) & D_3(z,y) & D_2(z,x,y) \end{bmatrix}, \qquad (13)
$$

where

$$
\begin{aligned}
D_2(x,y,z) &= \sqrt{\alpha}\frac{\partial^2}{\partial x^2}+\sqrt{\beta}\frac{\partial^2}{\partial y^2}+\sqrt{\beta}\frac{\partial^2}{\partial z^2}\\
D_3(x,y) &= (\sqrt{\alpha}-\sqrt{\beta})\frac{\partial^2}{\partial x\partial y}
\end{aligned}
$$

yields an equivalent state space model for (3). This L is a 3×3 differential operator, which is well-posed and realizable; hence, it can be used to simulate velocity fields that are tuned to the underlying physics by selecting appropriate α and β parameters. The use of similar well-posed models in simulations was demonstrated in 2-D in [10].

In the special case of $\alpha = \beta$ i.e., when the divergence and the curl components of the velocity field are penalized equally), the above L has the following compact form : $L = I_3 \otimes \nabla^2$, where the symbol $\otimes$ stands for the Kronecker product. In this case, the velocity components themselves are modeled as the solution of Poisson's equation driven by independent white noise processes, and can be simulated for arbitrary choice of driving noise processes.

IV. Conclusion

In this paper we considered state-space models for optical flow algorithms in 3-D. We showed that for the first-order DIV-CURL spline, although it is possible

to find a linear state-space model which has the same LMMSE solution as the variational solution, no well-posed model of the form (6) can be found. In order to develop well-posed models for 3-D OF, we use the second-order DIV-CURL splines and show the resulting state-space model.

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