FAST SURFACE-BASED MEASUREMENTS USING FIRST EIGENFUNCTION OF THE LAPLACE-BELTRAMI OPERATOR: INTEREST FOR SULCAL DESCRIPTION

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ABSTRACT

In this paper we propose a fast method to compute the longitudinal extension of surfaces using the extrema of the first eigenfunction of Laplace-Beltrami Operator and the *hot spots conjecture*. We also propose an original definition of the surface width based on the distance to the longest geodesic. We show that the implementation of our new definition of length is consistent with the one computed from brute force and that the time complexity is considerably improved. We have tested the numerical efficiency of our approach on simple simulations and applied it to cortical surface patches from a real MRI dataset. Besides our approach enriches global descriptors of sulci shapes with a third dimension : length, depth and now width.

Index Terms— Brain Morphometry, Laplace-Beltrami Operator, hot spots conjecture, Computational anatomy

1. INTRODUCTION

In medical shape analysis and computational anatomy there is a considerable need for morphometric descriptors, from very global and simple ones such as the volume [1] to more detailed ones such as eigenvalue spectrum [2]. These descriptors are biomarkers that can play an important part in diseases characterization [1] but also in quantitative description and modeling of brain development [3]. In that matter, cortical and sulcal morphometry are currently an active field of investigation [4].

In this work we revisit the simple but essential notion of length by proposing a fast and robust approach to extract the longest geodesic on a surface of genus zeros with boundaries. For this we rely on a mathematical conjecture based on the extrema of the first eigenfunction of the Laplace-Beltrami operator [5]. We also further propose a complementary surface descriptor similar to a width. Our goal should be distinguished from finding very detailed shape descriptors such as in [6] with inner distance. A direct application is proposed to enrich object-based sulcometry.

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2. OVERVIEW ON LAPLACE-BELTRAMI OPERATOR

2.1. Spectral theory

We recall some classical results that can be found in [7]. Given a compact Riemannian manifold $(\mathcal{M}, g)$, where g is a metric tensor, we introduce $L^2(\mathcal{M}) = \{u : \mathcal{M} \to \mathbb{R} / \int_{\mathcal{M}} u^2 < +\infty\}$ and the scalar product on $L^2(\mathcal{M})$, $< u, v >= \int_{\mathcal{M}} uv$. The spectrum of the *Laplace-Beltrami operator* (LBO) $\Delta_{\mathcal{M}} = \text{div} \circ \nabla_{\mathcal{M}}$ is discrete. We denote $\lambda_0 = 0 \leq \lambda_1 \leq ...\lambda_n \leq ...$ the *eigenvalues* of $-\Delta_{\mathcal{M}}$ and $\phi_0, \phi_1, ..., \phi_n, ...$ an associated orthonormal basis of *eigenfunctions* in $L^2(\mathcal{M})$ that satisfy:

$$-\Delta_{\mathcal{M}} \phi_n = \lambda_n \phi_n. \quad (1)$$

with boundary conditions on ∂M. In the following we will use Neumann boundary conditions, that is

$$g(\nabla_{\mathcal{M}} \phi_n, \mathbf{n}) = 0 \text{ on } \partial\mathcal{M} \quad (2)$$

In the discrete case it is possible to compute eigenfunctions of the Laplace-Beltrami operator on a mesh $\mathcal{M}_h$ that approximates $\mathcal{M}$ using the weak formulation of the eigenvalue problem and the finite elements method (FEM) [8]. This leads to a matrix eigenvalue problem:

$$[\nabla][U] = \lambda[M][U], \quad (3)$$

where $[U]$ is a vector representing an eigenfunction at each node of the mesh, $[\nabla]$ and $[M]$ are the stiffness and mass matrices whose expressions are given for instance in [9].

2.2. Hot Spots Conjecture

Several authors [9, 8, 10] have remarked that the first eigenfunction ϕ_1, also called *Fiedler vector*, allows to describe the

longitudinal extension of a shape. In [5], this observation has been linked to the *hot spots conjecture*, which is an active field of research in pure mathematics [11]. More precisely, authors in [5] propose a conjecture for closed surfaces (with no boundaries) of genus zero. In our case, we propose a similar conjecture for simply connected surfaces (no holes) with boundaries.

Conjecture 1. *Given a smooth simply connected surface $\mathcal{M}$ with boundary $\partial\mathcal{M} \neq \emptyset$, let us define $p_{min} = \arg\min_{p\in\mathcal{M}} \phi_1(p)$ and $p_{max} = \arg\max_{p\in\mathcal{M}} \phi_1(p)$. Then we have :*

(C1) $p_{min}, p_{max} \in \partial\mathcal{M}$

(C2) $\forall p, q \in \mathcal{M},\ d(p,q) \leq d(p_{min}, p_{max})$

where $d(p,q)$ corresponds to the geodesic distance (or shortest path distance) from p to q. $(C1)$ corresponds to the hot spots conjecture as formulated in [11] and $(C2)$ is the same expression as in [5]. It seems clear that (C2) implies (C1) so (C2) is a stronger result than (C1). We make use of this conjecture to define below the intrinsic length of a surface.

3. MEASUREMENTS OF SURFACES

We propose here to define canonical surface measurements that can be related to length or width for elementary surfaces like rectangles. These measurements will be independent of the surface position and intrinsic. This last property is not satisfied for instance if one wants to compute the length of a point cloud bounding box since two isometric surfaces can have different bounding box lengths (imagine for instance a bent rectangular sheet).

3.1. Definitions of surface length

We define three different approaches to compute an intrinsic length of a surface.

Definition 1. *Given a smooth surface $\mathcal{M}$, we define an intrinsic length by:*

$$l_1(\mathcal{M}) = \max_{p,q\in\mathcal{M}} d(p,q) \tag{4}$$

This length corresponds to the length of the longest path which can be obtained on $\mathcal{M}$. This length is also called diameter in the Riemannian geometry litterature [7]. On a triangular mesh, distance between points can be computed using Fast-Marching algorithm [12].

Definition 2. *Given a smooth surface $\mathcal{M}$, we define an intrinsic length by:*

$$l_2(\mathcal{M}) = d(p_{min}, p_{max}) \tag{5}$$

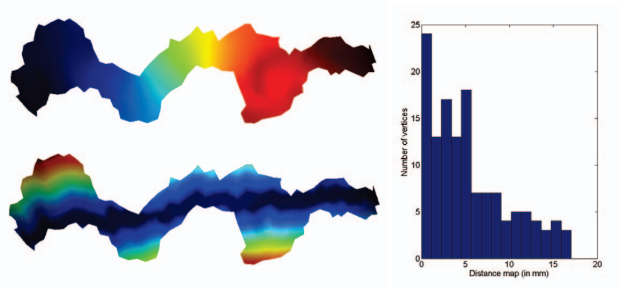


Fig. 1. Left top: first eigenfunction of a surface patch. Negative (resp. positive) values in blue (resp. red). Left bottom: distance map w to the longest geodesic obtained from Def. 2. Right: histogram of w on the boundary of the patch.

where p_{min} and p_{max} are the min and max of the Fiedler vector ϕ_1. To our knowledge this is a new definition of surface length. Conjecture 1 claims that $l_1(\mathcal{M}) = l_2(\mathcal{M})$.

Definition 3. *Given a smooth surface $\mathcal{M}$, we define an intrinsic length by:*

$$l_3(\mathcal{M}) = \frac{\pi}{\sqrt{\lambda_1}} \tag{6}$$

This definition can be found for instance in [9]. Intuitively, for a rectangle of length L and width w, the eigenvalues of Laplacian operator are given by $\lambda_{m,n} = \pi^2\left(\frac{m^2}{L^2} + \frac{n^2}{w^2}\right)$ with $m, n \geq 0$. the smallest non vanishing eigenvalue is therefore given by $\lambda = \frac{\pi^2}{L^2}$. By analogy we extend this definition to the case of connected surface patches with no holes.

3.2. Definition of surface width

In the following we call p_{min} (resp. p_{max}) the points used in definition 1 or 2.

Definition 4. *A longest geodesic $\mathcal{G}$ on $\mathcal{M}$ is a path $p : t \in [0,1] \rightarrow \mathcal{G} \subset \mathcal{M}$ such as $p(0) = p_{min}$, $p(1) = p_{max}$ and $d(p_{min}, p_{max}) = length(p)$*

There is not necessarily an unique longest geodesic: for instance imagine an half sphere with two opposite points on the equator. The path computation is performed through Fast-Marching algorithm and gradient descent of the distance map.

Once the longest geodesic is computed, the Fast-Marching algorithm is used to obtain the distance map w to $\mathcal{G}$. Then we consider the restriction of w on the boundary $\partial\mathcal{M}$. Next we can propose an original definition of surface width:

Definition 5. *We define a width by:*

$$w(\mathcal{M}) = 2 \oint_{\partial M} w(s)ds \Big/ \oint_{\partial M} ds \tag{7}$$

With this definition the width of a surface is obtained through the mean of the distribution w on the boundary ∂M .

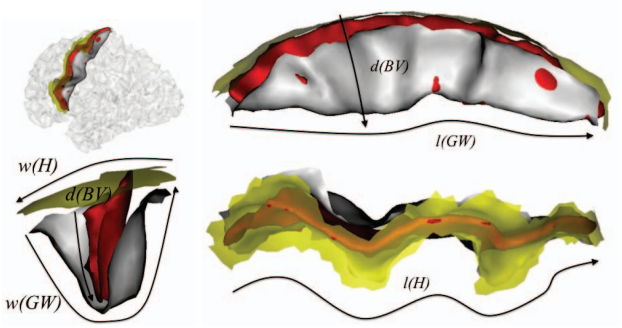


Fig. 2. Left top: Surface patches of central sulci on GW interface (gray), on Hull surface (yellow) and central sulcus (red). Right top, Left/right bottom: Three views of these objects with their descriptors.

4. RESULTS

4.1. Data and Preprocessing

Experiments were performed on a random set of 30 subjects from the ICBM database [13]. T1-weighted MR images were processed using the BrainVisa software [1] to get topologically spherical meshes of the left hemisphere hull (H) and grey/white matter interface (GW), with less than 30000 vertices. Using the Brainvisa/Surfpaint Toolbox [14] the sulcal basin of the central sulcus was semi-automatically delineated on the GW surface, then projected on the H surface to define a corresponding surface patch. Finally a mesh of the central sulcus and the corresponding depth profile were computed as defined in [4]. We call $d(BV)$ the maximum of this depth profile. Both GW and H surface patches, as well as the Central Sulcus mesh, are illustrated on Fig. 2.

4.2. Computation time

The Fast-Marching algorithm has been performed using Gabriel Peyre's toolbox [15] for Matlab with fast low-level functions in C++ using a heap structure. In this implementation the complexity is $O(N \log N)$ where N is the number of vertices. So in Definition 1 the complexity is $O(bN \log(N))$ where b is the number of vertices on the boundary. This number can be approximatively minored by $O(\sqrt{N})$ thanks to isoperimetric inequality [7] $p^2 \geq 4\pi A$ that links perimeter p and area A of a surface. So Definition. 1 implies at least $O(N^{3/2} \log N)$ operations.

With Definition 2 we use the eigenvalue problem (3) that can be solved with methods such as the Lanczos one implemented in [2]. Matrices building and extraction of the two first eigenvectors can be performed in $O(N)$ operations.

[1] http://www.brainvisa.info/

[2] Arnoldi Package, http://www.caam.rice.edu/software/ARPACK/

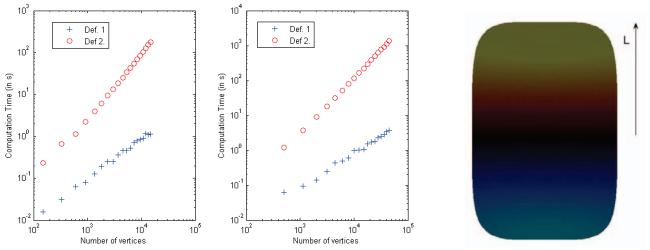


Fig. 3. Computation time as a function of the number of vertices for definitions 1. and 2. with $L = 1.5$ (left), $L = 5$ (middle). Simulated domain with $L = 1.5$ and its first eigenfunction (right).

This difference of complexity was tested on simulated surface patches: a rectangular-like planar shape was generated as the interior of the curve $|x/L|^5 + |y|^5 = 1$ with elongation parameter $L = 1.5$ and $L = 5$, as illustrated on Fig. 3. The number of vertices was varied between 10^2 and 5.10^4 and the computation time was measured for definition 1 and 2 on a 2.9 GHz dual core processor. Results are plotted in logarithmic scale on Fig. 3 and show the clear superiority of definition 2. For instance, with $L = 5$ and 45000 vertices, there is more than a factor of 300 between definition 1 ($\approx$ 1300 s) and 2 ($\approx$ 4 s).

4.3. Coherence of Length definitions

We have computed the length of GW and H surface patches. On Fig. 4 we present this length according to definition 2 and 3 with respect to the length according to definition 1. We observe that definition 1 and 2 coincide very well. For GW (resp. H), the mean relative error is 0.72% (resp 0.39%) , the maximum error is 3.05% (resp. 2.20%). We can also check that extrema in definitions 1 and 2 are well superimposed. For GW, $1/5$ correspond to same vertices and the error relatively to the longest geodesic is 10.43% while it becomes $1/2$ and 4.46% for Hull. Definition 3 and 1 have a good correlation (even if smaller), (0.91, $p < 0.001$) for GW and (0.68, $p < 0.001$) for H even if they do not perfectly match. Definition 3 underestimates length for GW but overestimates for H.

4.4. Surfacic patches vs volumic sulci

Our approach allows to add two relevant descriptors to cortical sulci: GW width and Hull width. However their coherence can be assessed via their correlation to another parameter such as maximum sulcal depth $d(BV)$.

$$w(GW)/2 = w(H)/2 + d(BV) \tag{8}$$

We observe a good correlation (0.76, $p < 0.001$) between $w(GW)/2$ and $w(H)/2 + d(BV)$, which is better than the correlation (0.46) between $w(GW)$ and $w(H)$, and the one

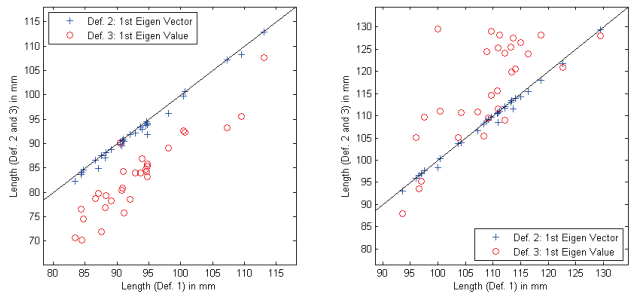


Fig. 4. Coherence between 3 length definitions for GW surface patches (left), H patches (right), for the 30 subjects.

(0.65) between $w(GW)$ and $d(BV)$. So we can conclude that combining depth and Hull width adds information on the knowledge of GW width.

5. DISCUSSION AND CONCLUSION

Our results provide a satisfying empirical verification of hot spots conjecture, even if not exact because of numerical approximations in implementations of definitions 1 and 2. Pragmatically, the consistency of results suggest to favor the definition 2 which is much more faster on our real parcel dataset (factor 25). For databases of hundreds of subjects and around 60 surface patches for each hemisphere with higher spatial resolution (150000 vertices), we can expect computation time of less than one hour with definition 2 versus a few days with definition 1. At last we have extended global descriptors of cortical sulci with GW width and Hull width. This approach creates a bridge between object-based and surface-based sulcal models, suggesting to combine measurements on GW interface, Hull surface and sulcal imprint itself to give a full 3D description of global sulcal size. This complementary descriptor shows to conceal new information and for instance, preliminary results of on going research suggest that length, depth and width can show non-proportional inter-subject variations.

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